Chapter One
Sentential Logic with 'if' and 'not'

1 SYMBOLIC NOTATION

In this chapter we begin the study of sentential logic. We start by formulating the basic part of the symbolic notation mentioned in the Introduction. For purposes of this chapter and the next, our symbolic sentences will consist entirely of simple sentences, called atomic sentences, together with molecular sentences made by combining simpler ones with connectives. The simple sentences are capital letters, which can be thought of as abbreviating sentences of English, as in the Introduction. In this chapter, the connectives are the negation sign, '~', and the conditional sign, '→'.

> The **negation sign**, '~', is used much as the word '*not*' is used in English, to state the opposite of what a given sentence says. For example, if 'P' abbreviates the sentence '*Polk was a president*', then '~P' abbreviates the sentence '*Polk was not a president*'.
>
> The **conditional sign**, '→', is used much as '*if . . , then . . .*' is used in English. If 'P' abbreviates the sentence '*Polk was a president*' and 'W' abbreviates '*Whitney was a president*' then '(P→W)' abbreviates the sentence '*If Polk was a president, then Whitney was a president*'.

We need to be precise about exactly what the **symbolic sentences** of Chapter 1 are:

- Any capital letter between 'P' and 'Z' is a symbolic sentence.
- If □ is a symbolic sentence, so is ~□
- If □ and ○ are symbolic sentences, so is (□→○).

Nothing is a symbolic sentence of Chapter 1 unless it can be constructed by means of these provisions.

Some terminology:

- A symbolic sentence containing no connectives at all is an **atomic** sentence. In this chapter and the next, only sentence letters are atomic.
- Any symbolic sentence that contains one or more connectives is called a **molecular** sentence.
- We call '~□' the **negation of** '□'.
- We call any symbolic sentence of the form '(□ → ○)' a **conditional** sentence; we call '□' the **antecedent** of the conditional, and '○' the **consequent** of the conditional.

Examples of symbolic sentences with minimal complexity are:

U
~U
(U→V)

The first is an atomic sentence. The second is the negation of that atomic sentence. The last is a conditional whose antecedent is the atomic sentence 'U' and whose consequent is the atomic sentence 'V'.

Once a molecular sentence is constructed, it can itself be combined with others to make more complex molecular sentences:

~(U→V)	*it is not the case that if U then V*
(~V → (U→V))	*if it is not the case that V then if U then V*
~~(V → (U→V))	*it is not the case that it is not the case that if V then if U then V*

The formation rules determine when parentheses occur in a symbolic sentence. When adding a negation sign to a sentence you do not add any parentheses. These are not symbolic sentences because they contain extra (prohibited) parentheses :

$\sim(U)$, $\sim(\sim U)$, $\sim((U\rightarrow V))$

Although '$\sim(U\rightarrow V)$' has a parenthesis immediately following the negation sign, that parenthesis got into the sentence when constructing '$(U\rightarrow V)$', and not because of the later addition of the negation sign.

When combining sentences with the conditional sign, parentheses are required. For example, this is not a sentence:

$U \rightarrow V \rightarrow W$

There is one exception to the need for parentheses. If a sentence appears all by itself, not as part of a larger sentence, then its outer parentheses may be omitted. So these sentences are taken informally to be conditional symbolic sentences:

$U\rightarrow V$
$\sim U \rightarrow V$
$(U\rightarrow V) \rightarrow \sim U$

> For purposes of Chapter 1:
>
> A sentence is in **official notation** if it can be constructed by using the processes given in the box above.
>
> It is in **informal notation** if it can be put into official notation by enclosing it in a single pair of parentheses.
>
> Anything that is not in either official notation or informal notation is not a sentence at all.

Any well-formed sentence can be "parsed" into its constituents. You begin with the sentence itself, and you indicate below it how it is constructed out of its constituents. First you locate the **main connective**, which is the last connective introduced when constructing the sentence. If the sentence is a negation, the main connective is the negation sign; you draw a vertical line under it and write the part of the sentence to which the negation sign is applied. If it is a conditional, the main connective is the conditional sign; you draw branching lines below the main conditional sign and write the antecedent and consequent:

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 ~P              P → Q
 |              /     \
 P             P       Q
```

If the parts are themselves complex, the parsing may be continued:

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   ~~(P → Q)          P → ~(Q→R)          (P→Q) → (~R→S)
   |                  /      \               /       \
   ~(P→ Q)           P      ~(Q→R)        P→Q       ~R→S
   |                          |          /   \      /   \
  P → Q                     Q → R       P     Q   ~R     S
  /    \                    /   \                  |
 P      Q                  Q     R                 R
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EXERCISES

1. For each of the following state whether it is a sentence in official notation, or a sentence in informal notation, or not a sentence at all. If it is a sentence, parse it as indicated above.

a. ~~~P
b. ~Q→~R
c. ~(Q~→R)
d. ~(~P)→~R
e. (P→Q) → (R→~Q)
f. P → (Q→R) → Q
g. (P → (Q→R) → Q)
h. (~S→R) → ((~R→S) → ~(~S→R))
i. P → (Q→P)

2 MEANINGS OF THE SYMBOLIC NOTATION

The negation sign: The logical import of the negation sign is this: it makes a sentence that is false if the sentence to which it is prefixed is true, and true if the sentence to which it is prefixed is false. It is common to summarize this behavior of the negation sign by means of what is called a **truth table**:

□	~□
T	**F**
F	**T**

In this table, 'T' stands for 'true' and 'F' for 'false'. Reading across the rows, the table says that when a sentence '□' is true, its negation, '~□' is false, and when a sentence '□' is false, its negation '~□' is true. When a connective can be defined by a truth table in this way, the connective is said to be "truth functional". This means that the truth value of the whole sentence that is created by combining this connective with another sentence is completely determined by ("is a function of") the truth value of the sentence with which it is combined. All connectives in our logical notation will be truth functional.

The negation sign corresponds naturally to either of two locutions in English. One is the locution '*it is not the case that*' placed on the front of a sentence. The other is the word '*not*' when used within a simple sentence. So if 'S' is taken to abbreviate '*The salad was tasty*', these have the same logical content:

> ~S
> ***It is not the case that*** *the salad was tasty*
> *The salad was* ***not*** *tasty*

In the logical tradition, the locution '*fail to*' is often taken to have the meaning of negation. This is limited to certain uses of English. For example, if you say *Samantha failed to reach the summit* you are probably reporting a situation in which she tried to reach the summit, but did not reach it. However, if you say that the number of buttons on a shirt fail to match the number of buttonholes, you are probably not saying that the buttons tried to match the buttonholes but did not match them, you may only be saying that the buttons and buttonholes do not match. In this usage, '*fail to*' may report only negation. The sentence:

> *The salad* ***failed to*** *be poisonous*

then reports the negation of the sentence '*The salad was poisonous*', and it may be symbolized '~S'. In the exercises we will assume that '*fail to*' only amounts to negation.

The conditional sign: The conditional sign is meant to capture some part of the logical import of '*if . . , then*' in English. But it is not completely clear under what circumstances an '*if . . , then*' claim in English is true. It seems clear that any English sentence of the form '*If P then Q*' is false when '*P*' is true but '*Q*' is false. If you say '*If the Angels win there will be a thunderstorm*', then if the Angels do win and if there is no thunderstorm, what you said is false. In other cases things are not so clear. Consider these conditional sentences uttered in normal circumstances:

> *If it rains, the game will be called off.*
> *If the cheerleaders are late, the game will be called off.*

Now suppose that it rains, and the cheerleaders are late, and the game is called off. Are the sentences above true or false? Most people would be inclined to say that the first is true. But the second is less obvious. After all, the game was not called off *because* the cheerleaders were late. So there is something funny about the second sentence. If it is false, it will be impossible to capture the logical import of conditionals by means of any truth functional connective. For the truth of the first sentence above requires that some conditionals be true when both their parts are true, and the second would require that some conditionals be false when both their parts are true.

However, you might hold that the second sentence above is true. Granted, the game was not called off *because* the cheerleaders were late, but so what? The second sentence doesn't say anything at all about *why* the game was called off. It only says that it will be called off if the cheerleaders are late; and they *were* late, and the game *was* called off, so it is true. If so, perhaps conditionals *are* truth functional.

There is no universal agreement about how conditionals work in natural language. The position taken in this text is that '*if . . , then. . .*' is *sometimes* used to express what is called the "material conditional". This is the use of '*if . . , then . . .*' where a conditional sentence is false in case the antecedent is true and the consequent false, and it is true in every other case. This use is truth functional. It is described by means of this truth table:

□	○	□→○
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

The conditional is used in this way by mathematicians, and by others. We will assume in doing exercises and examples that the logical import of '*if . . , then*' is intended to coincide with our symbolic '→'. There may be other uses of '*if . . , then*' that convey more than '→', but we will not address them in this text.

The word '*if*' in English has many synonyms. In at least some contexts these are all interchangeable:

if	***If*** *Maria sings, Xavier will leave*
provided that	***Provided that*** *Maria sings, Xavier will leave*
assuming that	***Assuming that*** *Maria sings, Xavier will leave*
given that	***Given that*** *Maria sings, Xavier will leave*
in case	***In case*** *Maria sings, Xavier will leave*
on the condition that	***On the condition that*** *Maria sings, Xavier will leave*

Using 'S' for '*Maria sings*' and 'X' for '*Xavier will leave*', these can all be symbolized as

S→X

'*If*' clauses in English may also occur at the end of a sentence instead of at the beginning. So these also may be symbolized as 'S→X':

Xavier will leave ***if*** *Maria sings*
Xavier will leave ***provided that*** *Maria sings*
Xavier will leave ***assuming that*** *Maria sings*
Xavier will leave ***given that*** *Maria sings*
Xavier will leave ***in case*** *Maria sings*
Xavier will leave ***on the condition that*** *Maria sings*

In either use, the word '*If*' immediately precedes the antecedent of the conditional.

Many of these "synonyms" of '*if*' can be used to say more than what is said with a simple use of the word '*if*'. For example, a person who says '*assuming that*' may want to convey that s/he is indeed making a certain assumption, and not just saying '*if*'. But in other contexts no assuming is indicated. A physicist who says '*Assuming that there are additional planets with orbits outside the orbit of Pluto, we will need to send space probes to investigate them*' may simply be responding to the question '*What if there are planets beyond Pluto?*', and not doing any assuming at all. In doing the exercises we will take for granted that the locutions identified above are being used in the most minimal sense of '*if*', which we take to be that of the connective '→'.

Only if: The word '***only***' can be added to the word '*if*', to make '***only if***'. The '*only*' has the effect of reversing antecedent and consequent. As a result, whereas '*if*', when used alone, immediately precedes the antecedent of a conditional, '*only if*' immediately precedes the consequent. So we have these equivalences:

If P, Q	P→Q
Only if P, Q	Q→P
P if Q	Q→P
P only if Q	P→Q

Some will find it more natural to represent 'P only if Q' by 'If not Q then not P', or '~Q → ~P'. It will turn out that this is logically equivalent to 'P → Q'. We will generally use the latter form because it's simpler.

When '*only if*' comes first, there are grammatical changes in the last clause, which converts into its interrogative word order:

<u>The game will be called off</u> only if it rains = Only if it rains <u>will the game be called off</u>

'*Only*' may also precede any of the synonyms of '*if*', so these all may be symbolized as 'X → S':

*Xavier will leave **only if** Maria sings*
*Xavier will leave **only provided that** Maria sings*
*Xavier will leave **only assuming that** Maria sings*
*Xavier will leave **only given that** Maria sings*
*Xavier will leave **only in case** Maria sings*
*Xavier will leave **only on the condition that** Maria sings*

Only *if* *Maria sings will Xavier leave*
Only provided that *Maria sings will Xavier leave*
Only assuming that *Maria sings will Xavier leave*
Only given that *Maria sings will Xavier leave*
Only in case *Maria sings will Xavier leave*
Only on the condition that *Maria sings Xavier will leave*

EXERCISES

For these exercises assume that 'S' abbreviates '*Susan will be late*' and 'R' abbreviates '*It will rain*'.

1. For each of the following sentences say which symbolic sentence is equivalent to it.

a. Only if it rains will Susan be late
S→R
R→S

b. Susan will be late provided that it rains
S→R
R→S

c. Susan won't be late
~S
~~S

d. Susan will be late only if it rains
S→R
R→S

e. Given that it rains, Susan will be late
S→R
R→S

2. Symbolize each of the following:

a. Susan will be late only provided that it rains
b. Only on condition that it rains will Susan be late
c. Susan will be late only in case it rains
d. Susan will be late only if it rains
e. It is not the case that Susan will be late

3 SYMBOLIZATION: TRANSLATING COMPLEX SENTENCES INTO SYMBOLIC NOTATION

In representing simple sentences of English by sentential letters you need to say which letter abbreviates which sentence. So far this has been done informally, by choosing sentential letters that are already used in a prominent place in the English sentence, as in using 'S' for '*Sally will be late*'. But if different English sentences share their prominent letters, we can soon run out of natural sentential letters to choose. So instead we give a **scheme of abbreviation**, which pairs off sentence letters with the English sentences that they abbreviate. An example is:

X *Susan will be late*
Y *It will rain*

With this scheme we would represent '*Given that it rains, Susan will be late*' by:

$Y \rightarrow X$

Any way of "symbolizing" English sentences in logical notation, or of "translating" English into logical notation, is done relative to a scheme of abbreviation. A translation or symbolization that is correct on one scheme may be incorrect on others.

Complex sentences of English generally translate into complex sentences of the logical notation. Here it is important to be clear about the grouping of clauses in the English sentence. Consider, the sentence:

If Roberta doesn't call, Susan will be distraught

This is a conditional whose antecedent is a negation. Using 'P' for '*Roberta calls*' and 'Q' for '*Susan will be distraught*', this may be symbolized:

$\sim P \rightarrow Q$

It is not the negation of a conditional:

$\sim(P\rightarrow Q)$

To make the negation of a conditional, you need to say something like:

It is not the case that if Roberta calls, Susan will be distraught

which is symbolized as:

$\sim(P \rightarrow Q)$

There are a few fundamental principles that govern symbolizations of English sentences in the logical notation.

> SOURCES OF '~'
>
> The locution '*fail to*' always yields a negation sign that applies to the symbolization of the smallest sentence that '*fail to*' is part of.
>
> Likewise for the word '*not*'.
>
> The expression '*it is not the case that*' applies to a sentence immediately following it.

Notice that in the sentence '*It is not the case that Willa will leave if Sam does*' there are two grammatical sentences that immediately follow '*It is not the case that*', namely, '*Willa will leave*' and '*Willa will leave if Sam does*'. So there are two ways to symbolize the sentence:

$S \rightarrow \sim W$
$\sim(S\rightarrow W)$

The sentence is in fact ambiguous.

SOURCES OF '→'

If: The word '*if*' always gives rise to a conditional, '□→○'.
Wherever '*if*' occurs (not as part of '*only if*'), the antecedent of the conditional is the symbolization of a sentence immediately following '*if*.
The consequent of the conditional is either the symbolization of a sentence immediately preceding '*if*' (with no comma in between) -- as in '□ *if* ○' -- or it is the symbolization of a sentence immediately following the sentence that is symbolized as the antecedent -- as in '*if* □ *then* ○'.

Then: If '*then*' occurs it must be paired with a preceding '*if*'.
The antecedent of the conditional introduced by '*if*' is the symbolization of the sentence exactly between '*if*' and '*then*'.
Its consequent is the symbolization of a sentence immediately following '*then*'.

Only if: The expression '*only if*' always gives rise to a conditional, '□→○'.
The consequent of '□→○' is the symbolization of a sentence immediately following '*only if*' -- as in '○ *only if* □' -- or as in '*only if* ○, □'.
The antecedent of '□→○' is the symbolization of a sentence immediately preceding '*only if*' (with no comma in between) -- '□ *only if* ○' -- or of a sentence immediately following the antecedent -- as in '*only if* ○, □'. In the latter case, that sentence is grammatically changed (to its interrogative word order).

Illustration: These principles determine that the sentence: '*Pat won't call only if it is not the case that the quilt is dirty*' is symbolized:

$$\sim P \rightarrow \sim Q$$

The (contracted) '*not*' in '*Pat won't call*' yields a negation that applies directly to 'P'. The '*it is not the case that*' yields a negation that applies directly to 'Q', since '*the quilt is dirty*' is the only sentence immediately following '*it is not the case that*'. The only sentence immediately to the left of the '*only if*' is '*Pat won't call*', so that is the antecedent of the conditional, and the only sentence immediately to the right of the '*only if*' is '*it is not the case that the quilt is dirty*', so that is the consequent.

Illustration: In the sentence '*If Wilma leaves then Xavier stays if Yolanda sings*' the first '*if . , then . . .*' exactly encloses '*Wilma leaves*', so W is the antecedent of the first conditional. There are two sentences immediately following '*then*'; they are the whole '*Xavier stays if Yolanda sings*' and just '*Xavier stays*'. So the sentence must have the form:

W → (*Xavier stays if Yolanda sings*)

or

(W → X) *if Yolanda sings*

The second '*if*' comes between its consequent and antecedent. It must give rise to a conditional that has 'Y' as its antecedent, since the only sentence following the '*if*' is '*Yolanda sings'*. The consequent of that conditional can be the symbolization of either just '*Xavier stays*', or '*If Wilma leaves then Xavier stays*', since each of these immediately precedes the '*if*'. The first of these fits with the first partial symbolization above, giving:

$$W \rightarrow (Y \rightarrow X)$$

and the second fits with:

$$Y \rightarrow (W \rightarrow X).$$

Both of these symbolizations are possible, which agrees with the intuition that the original English sentence is ambiguous. (Some people find the first reading more natural than the second, but he second is a possible reading under some circumstances.)

Illustration: In the sentence '*If Wilma leaves then Xavier stays only if Yolanda sings*' the first '*if*' is exactly as in the previous case, so the sentence has the form:

W → (*Xavier stays only if Yolanda sings*)

or

(W → X) *only if Yolanda sings*

The '*only if*' comes between its antecedent and consequent. It must give rise to a conditional that has 'Y' as its consequent, since the only sentence following the '*only if*' is '*Yolanda sings*'. The antecedent of that conditional can be the symbolization of either just '*Xavier stays*', or '*If Wilma leaves then Xavier stays*', since each of these immediately precedes the '*only if*'. The first of these fits with the first partial symbolization above, giving:

W → (X→Y)

and the second fits with:

(W→X) → Y

Again, the sentence is ambiguous.

The principles above also apply when '*if*' is replaced by one of its synonyms, such as '*given that*'.

So '*If Wilma leaves then Xavier stays provided that Yolanda sings*' has the same symbolization options as '*If Wilma leaves then Xavier stays if Yolanda sings*':

W → (Y→X)

and

Y → (W→X).

Commas: We have seen that the fundamental principles governing words that yield negations and conditionals can permit a significant amount of ambiguity. A common way to eliminate such ambiguity from sentences is to use commas to indicate how parts of the symbolization are to be grouped. Commas are used for a wide variety of purposes, so the presence of a comma may be irrelevant to the symbolization. But sometimes they are used to indicate that sentences should be grouped together, that is, combined into a single sentence. When this happens, the comma appears right after the sentence that results from the grouping. Or, a comma may be used to indicate that sentences to the right should be grouped together.

COMMAS A comma indicates that the symbolizations of sentences to its left should be combined into a single sentence, or that sentences to its right should be combined into a single sentence.

For example, above we saw that the sentence:

If Wilma leaves then Xavier stays if Yolanda sings

has two possible symbolizations:

W → (Y→X)

and

Y → (W→X).

If a comma appears before the first '*then*':

If Wilma leaves, then Xavier stays if Yolanda sings

this forces the first symbolization, since it requires that 'X' and 'Y' be grouped together. But if the comma appears later:

If Wilma leaves then Xavier stays, if Yolanda sings

this forces the second symbolization, since now 'W' and 'X' must be grouped together.

Sometimes there are no phrases to be kept together, as in:

If Wilma leaves, then Xavier stays

There is only one sentence to the left of the comma, and only one to the right, so the comma is redundant; we just have:

$W \rightarrow X$

EXERCISES

For these questions please use the following scheme of abbreviation:

V	*Veronica will leave*
W	*William will leave*
Y	*Yolanda will leave*

1. For each of the following say which of the proposed translations are correct.

a. *If Veronica doesn't leave William won't either*
 $\sim(V \rightarrow W)$
 $\sim V \rightarrow \sim W$
 $V \rightarrow \sim\sim W$

b. *William will leave if Yolanda does, provided that Veronica doesn't*
 $(W \rightarrow Y) \rightarrow \sim V$
 $V \rightarrow (W \rightarrow Y)$
 $\sim V \rightarrow (Y \rightarrow W)$

c. *If Yolanda doesn't leave, then Veronica will leave only if William doesn't*
 $\sim Y \rightarrow (\sim W \rightarrow V)$
 $\sim Y \rightarrow (V \rightarrow \sim W)$
 $\sim W \rightarrow (\sim Y \rightarrow V)$

d. *If Yolanda doesn't leave then Veronica will leave, given that William doesn't*
 $\sim Y \rightarrow (\sim W \rightarrow V)$
 $\sim Y \rightarrow (V \rightarrow \sim W)$
 $\sim W \rightarrow (\sim Y \rightarrow V)$

2. For each of the following produce a correct symbolization

a. *William will leave if Veronica does*

b. *Veronica won't leave if William does*

c. *If Veronica leaves, then if William doesn't leave, Yolanda will leave*

d. *If Veronica doesn't leave if William doesn't, then Yolanda won't.*

e. *William won't leave provided that Veronica doesn't leave*

f. *If William leaves, then if Veronica leaves so will Yolanda*

g. *William will leave only if if Veronica leaves then so will Yolanda*

h. *William will leave only if Veronica leaves, only provided that Yolanda will leave*

4 RULES

A "rule" is a particular valid form of argument which may be used in extended reasoning. Certain rules involving negations and conditionals have been recognized for centuries, and they have traditional names. The basic rules used in this chapter are:

RULES

Repetition:	□		
	∴ □		
Modus Ponens:	□→○		
	□		
	∴ ○		
Modus Tollens:	□→○		
	~○		
	∴ ~□		
Double negation:	□	or	~~□
	∴ ~~□		∴ □

Repetition is the most trivial rule; it indicates that if you have any sentence you may validly infer it from itself. Although trivial, this rule will have an important use later when we construct a method of showing that an argument is valid.

Modus ponens indicates that if you have any conditional sentence along with its antecedent you may infer its consequent. For example, this valid argument is an instance of modus ponens:

If Polk was a president, so was Whitney	P→W
Polk was a president	P
∴ *Whitney was a president*	∴ W

This rule may be justified by noting that a conditional with a true antecedent and false consequent is false.

Modus Tollens indicates that if you have any conditional sentence along with the negation of its consequent, you may infer the negation of its antecedent. For example, this valid argument is an instance of modus tollens:

If Polk was a president, so was Whitney	P→W
Whitney wasn't a president	~W
∴ *Polk wasn't a president*	∴ ~P

This rule may be justified in a similar way to that used in justifying modus ponens.

Double negation indicates that from any sentence you may infer the result of putting two negation signs on the front, or vice versa. For example, both of these valid arguments are instances of double negation:

Polk was a president	P
∴ *It is not the case that Polk wasn't a president*	∴ ~~P

It is not the case that Polk wasn't a president	~~P
∴ *Polk was a president*	∴ P

It should be obvious upon reflection that any argument whose conclusion follows from its premises by a single application of one of these rules is formally valid. That is, there cannot be a situation in which it has true premises and a false conclusion.

These rules apply to anything that fits their pattern, even if it is complex. For example, this is an instance of modus ponens:

	If Polk was a president, Whitney wasn't		$P \to \sim W$
	Polk was a president		P
∴	*Whitney wasn't a president*	∴	$\sim W$

And so is this:

	If Polk was a president then if Roosevelt was a president, so was Truman	$P \to (R \to T)$
	Polk was a president	P
∴	*If Roosevelt was a president then so was Truman*	$R \to T$

The following is <u>not</u> an instance of modus ponens:

	If Whitney was a president, so was Truman		$W \to T$
	Truman was a president		T
∴	*Whitney was a president*	∴	W

This argument is in fact invalid. It is an instance of a famous fallacy called "affirming the consequent".

Likewise, the following is <u>not</u> an instance of modus tollens:

	If Whitney was a president, so was Truman		$W \to T$
	Whitney wasn't a president		$\sim W$
∴	*Truman wasn't a president*	∴	$\sim T$

This too is a famous fallacy, called "denying the antecedent".

EXERCISES

1. For each of the following arguments, say whether it is an instance of modus ponens, or modus tollens, or double negation, or none of the above.

a. $P \to \sim Q$; Q; ∴ $\sim P$

b. $\sim P \to Q$; ∴ $\sim(P \to Q)$

c. $\sim\sim(P \to Q)$; ∴ $P \to Q$

d. $\sim P \to \sim Q$; $\sim P$; ∴ $\sim Q$

e. $\sim P \to \sim Q$; $\sim\sim Q$; ∴ $\sim\sim P$

f. $P \to Q$; $\sim R$; ∴ $\sim P$

g. $P \to (R \to Q)$; P; ∴ $R \to Q$

h. $P \to (R \to Q)$; $R \to \sim Q$; ∴ $\sim P$

i. $\sim\sim(P \to Q)$; $\sim Q$; ∴ $P \to Q$

2. Given the sentences below, say what, if anything, can be inferred in one step by modus ponens, or modus tollens, or double negation.

a. $\sim W \to \sim X$; $\sim W$; ∴ ?

b. $\sim W \to \sim X$; $\sim\sim X$; ∴ ?

c. $W \to X$; $\sim W$; ∴ ?

d. $W \to (R \to X)$; W; ∴ ?

e. $W \to (R \to X)$; $\sim R \to X$; ∴ ?

f. $\sim\sim(W \to X)$; $\sim X$; ∴ ?

g. $W \to \sim X$; X; ∴ ?

h. $\sim W \to X$; ∴ ?

i. $\sim\sim(W \to X)$; ∴ ?

5 DIRECT DERIVATIONS

Complex reasoning often consists of stringing together simple inferences so as to show the validity of a complex argument. Here is an example. We are given the following premises and conclusion:

If Polk was a president then if Whitney was a president so was Trump.
Polk wasn't a president only if Trump was a president.
Trump wasn't a president.
∴ *Whitney wasn't a president.*

We may reason as follows:

> Since Trump wasn't a president (given) and Polk wasn't a president only if Trump was (given), it is not the case that Polk wasn't a president. So Polk was a president. But if Polk was a president, then if Whitney was a president, so was Trump (given); so if Whitney was a president so was Trump. But Trump wasn't a president (given), so Whitney wasn't a president.

The pattern of reasoning is easier to follow if the sentences are given in symbolic form:

First premise:	$P\to(W\to T)$
Second premise:	$\sim P\to T$
Third premise:	$\sim T$
Conclusion:	∴ $\sim W$

The above reasoning can be systematized and explained using the following format, where each line contains a sentence followed by a "justification" – an explanation of why the sentence is there.

1. To show ~W:

"Since Trump wasn't a president (given) and Polk wasn't a president only if Trump was (given), it is not the case that Polk wasn't a president."

2.	$\sim T$	pr	*this line is a premise*
3.	$\sim P\to T$	pr	*this line is a premise*
4.	$\sim\sim P$	2 3 mt	*this line follows from 2 and 3 by modus tollens*

"So Polk was a president"

5.	P	4 dn	*this line follows from line 4 by double negation*

"But if Polk was a president, then if Whitney was a president, so was Trump (given); so if Whitney was a president so was Trump."

6.	$P\to(W\to T)$	pr	*this line is a premise*
7.	$W\to T$	5 6 mp	*this line follows from 5 and 6 by modus ponens*

"But Trump wasn't a president (given), so Whitney wasn't a president."

8.	$\sim T$	pr	*this line is a premise*
9.	$\sim W$	7 8 mt	*this line follows from 7 and 8 by modus tollens*

Lines 2-4 indicate that the first part of the reasoning appeals to two of the premises of the argument, and it draws a conclusion from them by modus tollens. Line 5 indicates that the reasoning goes from line 4 to 5 by double negation. Lines 6 and 7 indicate that line 5 together with the first premise lead to the sentence on line 7 by modus ponens. Finally, lines 8 and 9 indicate that line 7 together with the third premise lead to the conclusion of the argument.

This layout of premises and inferences constitute most of the ingredients of what we will call a derivation. Every line consists of a line number followed by a sentence followed by a justification. The sentence on each line either (i) occurs as a premise, and the line is justified by writing "pr", or (ii) follows from previous lines by a rule, and the line is justified by writing the number(s) of the line(s) from which it follows, along with a short name of the rule. The short names of the rules that we have so far are "r", "mp", "mt", and "dn".

The particular approach taken here is to see a derivation as carrying out a task. Each task is to show that a sentence follows from certain things. Our derivations begin with a special line stating the task; that is, stating what is to be shown. In the sequence of steps above it would come first, and would be of this form:

1. Show ~W

A "show" line may be introduced at any time, and it does not need a justification, because it only states what it is we intend to derive. All other lines need justifications.

Suppose a derivation is to be constructed, guided by the reasoning given above. Following the 'show' line we repeat two of the premises, justifying them with the notation 'pr':

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr

From these two lines we infer the third by modus ponens:

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr
4.	~~P	2 3 mt

Next, we write 'P', indicating that we are deriving it from line 3:

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr
4.	~~P	2 3 mt
5.	P	4 dn

Still following out the reasoning on the previous page, we repeat another premise, and then infer 'W → T' from it together with line 4 using modus ponens:

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr
4.	~~P	2 3 mt
5.	P	4 dn
6.	P→(W→T)	pr
7.	W→T	5 6 mp

Again, we insert the second premise, and infer '~W' from it and line 7 by modus tollens:

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr
4.	~~P	2 3 mt
5.	P	4 dn
6.	P→(W→T)	pr
7	W→T	5 6 mp
8.	~T	pr
9.	~W	7 8 mt

The reasoning ends here, where we have completed the task of showing '~W'. At this point we need a way to indicate that the task of showing '~W' has been completed. The completion of the task is indicated by writing "dd" after that line (meaning "direct derivation"); then the "Show" on the show line is cancelled by drawing a line through it -- because the task has been completed -- and the steps used in that process are boxed off. Specifically, since the sentence to be shown occurs on line 8, you may **write "dd" to its right, box all lines below the show line, and cancel the "Show"**:

1.	~~Show~~ ~W		*Cancel the "Show"*
2.	~T	pr	
3.	~P→T	pr	
4.	~~P	2 3 mt	
5.	P	4 dn	*Box the lines*
6.	P→(W→T)	pr	
7.	W→T	5 6 mp	
8.	~T	pr	
9.	~W	7 8 mt dd	*Write "dd"*

The cancellation of the show line indicates that the task was successfully completed, and the boxing encloses the lines used in completing that task.

It is also permissible to wait and write the "dd" on a later line. Such a line contains no sentence itself; its justification consists of the number of the line where the target sentence occurs, followed by "dd". Here is the same derivation with the dd justification on a later line:

1.	~~Show~~ ~W		
2.	~T	pr	
3.	~P→T	pr	
4.	~~P	2 3 mt	
5.	P	4 dn	
6.	P→(W→T)	pr	
7.	W→T	5 6 mp	
8.	~T	pr	
9.	~W	7 8 mt	
10.		9 dd	*Empty line with "dd"*

It is often a matter of taste which technique to use for indicating the completion of a direct derivation.

Here is another illustration of a direct derivation, used to show this argument valid:

Q→~S
V→X
~V→S
~X
∴ ~Q

The derivation begins with a line indicating that the task is to show the conclusion of the argument:

1. Show ~Q

The next few lines give the reasoning steps:

2.	V→X	pr
3.	~X	pr
4.	~V	2 3 mt
5.	~V→S	pr
6.	S	4 5 mp
7.	~~S	6 dn
8.	Q→~S	pr
9.	~Q	7 8 mt

On line 9 we have completed the task. So we write "dd" and box and cancel:

1.	~~Show~~ ~Q	
2.	V→X	pr
3.	~X	pr
4.	~V	2 3 mt
5.	~V→S	pr
6.	S	4 5 mp
7.	~~S	6 dn
8.	Q→~S	pr
9.	~Q	7 8 mt dd

(Line 7 is necessary before using modus tollens with line 8. This form of inference is indeed valid:

□→~○
○
∴ ~□

however, it is not itself an instance of modus tollens. It is instead an inference that is easily justified using double negation along with modus tollens.)

We indent all of the lines immediately following a show line; this is a device for keeping track (by indentation) of where the task that is initiated by the "show" is being carried out. The indentation also reserves a space for the box that will be drawn if the derivation is successful.

In getting precise about how to construct a direct derivation, it will help to specify what previously occurring things can be appealed to when applying a rule. We will say that a previous line is available from a given line just in case it is an earlier line that is not an uncancelled show line and is not already in a box:

> In a derivation from a set P of sentences, a line is **available from** a given line just in case it is a member of P or it is an earlier line that is not an uncancelled show line and is not already in a box.

Whether a line is available or not depends on your perspective. A show line is not available from the line immediately below it -- because it is not yet cancelled. But once it is cancelled, it is available from all lines below the line from which it was cancelled. And a line may be available from a given line, but once it is boxed, it is not available from any line below the line from which it was boxed.

> A **direct derivation** from a set of sentences P consists of a sequence of lines (including justifications when appropriate) that is built up, step by step, where each step is in accordance with these provisions:
>
> - A show line consists of the word "Show" followed by a sentence. The first step of producing a derivation must be to introduce a show line. A show line also may be introduced at any later step. Show lines are not given a justification.
> - At any step, any sentence from the set of sentences, P, may be introduced, justified with the notation "pr".
> - At any step a line may be introduced if it follows by a rule from previous available lines in the derivation; it is justified by citing the numbers of those previous lines and the name of the rule.
> - If a line is introduced whose sentence is the same as the sentence in the closest previous uncancelled show line, one may, as the next step, write "dd" at the end of that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
> - (Alternatively) At any step, if any previous available line contains a sentence that is the same as that in the closest previous uncancelled show line, one may introduce a line with no sentence on it, justifying it by citing the number of the earlier line followed by "dd"; one then draws a line through the word "Show", and draws a box around all the lines below that show line, including the current line.

These instructions show how to construct a derivation in a step by step fashion. Any number of steps results in a derivation, but not necessarily one that completes any of the tasks set out by its show lines. For example, when constructing a derivation above, at a certain stage we had reached this far in the process:

1.	Show ~W	
2.	~T	pr
3.	~P→T	pr
4.	~~P	2 3 mt

This sequence of lines satisfies the conditions for being a derivation as defined above. But there is a sense in which it is not yet finished. For this purpose we define a "complete" derivation:

> A derivation is **complete** if every show line is cancelled and every line that is not a show line is boxed.

The point of doing a derivation is often to show that a certain argument is formally valid. When a derivation shows that an argument is formally valid, we say that it "validates" the argument:

> A derivation **validates** an argument if and only if it is a complete derivation from the premises of that argument, and the conclusion of the argument appears on an unboxed cancelled show line in the derivation.

EXERCISES

1. Check through each line of the following direct derivations to determine whether it can be constructed by means of the provisions for direct derivations given above, where the set P is taken to be the premises of the displayed arguments. (When assessing a given line, assume that all previous lines are correct.)

Argument: P → (Q→~R)
~P → ~Q
Q
∴ ~R

1.	~~Show~~ ~R	
2.	Q	pr
3.	~~Q	2 dn
4.	~P → ~Q	pr
5.	~~P	3 4 mt
6.	P	6 dn
7.	P → (Q→~R)	pr
8.	Q→~R	6 7 mp
9.	~R	2 8 mp dd

Argument: P → (R→~Q)
~P → ~Q
Q
∴ ~R

1.	~~Show~~ ~R	
2.	Q	pr
3.	~P→~Q	2 pr
4.	P	2 3 mt
5.	R→~Q	4 mp
6.	~~Q	2 dn
7.	~~R	5 6 mt
8.	R	7 dn
9.	~Q	5 8 mp dd

Argument: ~O
S → (W→~O)
O → S
W
∴ ~S

1.	~~Show~~ ~S	
2.	W	pr
3.	~S	pr
4.	O → S	pr
5.	~O	3 4 mt
6.	W	pr
7.	~(W→~O)	5 6 mt
8.	S → (W→~O)	pr
9.	~S	7 8 mt
10.		9 dd

2. Construct direct derivations to validate each of the following arguments:

P
Q → ~P
R → Q
∴ ~R

W → ~(V→~Y)
X → (V→~Y)
V → Y
(V→Y) → X
∴ ~W

(W→Z) → (Z→W)
(Z→W) → ~X
P → X
~~P
∴ ~(W→Z)

6 CONDITIONAL DERIVATIONS

In this section we learn about one of the most powerful and useful procedures for constructing proofs in a natural way. The procedure is called conditional derivation. It is meant to reflect a natural reasoning process that involves *hypothetical inference*. Suppose that you wish to show that the following argument is valid:

If Robert drives, Sam won't drive.
If Sam doesn't drive, Teresa won't go.
Willa will go only if Teresa does.
∴ *If Robert drives, Willa won't go.*

If we try to reason as above using our rules mp, mt, and dn, we will not succeed; none of them apply to the premises we are given. What you would probably do on your own is to reason somewhat as follows:

ASSUME that Robert drives
Well, if he drives, Sam won't (given); so Sam won't drive
But if Sam doesn't drive, Teresa won't go (given), so Teresa won't go.
But Willa will go only if Teresa does (given), so Willa won't go.
So, SUMMING UP, if Robert drives, Willa won't go.

The middle three steps look familiar; they are inferences from premises and previously stated sentences, and they are all justifiable by rules that we have. But the first and last steps are new. What does it mean to "assume", as we have done in the first step, and what is this "summing up" in the last step? What role do these have as legitimate parts of a piece of reasoning?

Here is what goes on in "conditional" reasoning. Our goal is to show that a certain conditional sentence follows from certain premises. (In the example above, the conditional sentence is '*If Robert drives, Willa won't go*'.) We then "assume" the antecedent of the conditional. If we can use this to derive the consequent of the conditional, we conclude that this reasoning has shown the conditional itself to follow from the given premises. An example:

R→~S
~S→~T
W→T
∴ R→~W

A derivation using the conditional derivation technique begins with a line specifying the task, which is to show the conclusion. This is followed by an assumption of the antecedent of the conditional to be shown:

1.	Show R→~W	
2.	R	ass cd ← *assumption for conditional derivation (the goal is now to derive the consequent: ~W)*

The reasoning in the center of the derivation proceeds normally:

3.	R→~S	pr
4.	~S	2 3 mp
5.	~S→~T	pr
6.	~T	4 5 mp
7.	W→T	pr
8.	~W	6 7 mt

Now that we have derived the consequent of the conditional on line 8, We cite "cd", and we box and cancel:

1.	~~Show~~ R→~W		*Cancel the "Show"*
2.	R	ass cd	
3.	R→~S	pr	
4.	~S	2 3 mp	
5.	~S→~T	pr	*Box the lines*
6.	~T	4 5 mp	
7.	W→T	pr	
8.	~W	6 7 mt cd	*Write "cd"*

Lines 2-8 show that given the premises, we may derive '~W' from 'R'. Our conditional derivation technique says that this amounts to showing that those premises validate 'R→~W', so we may box and cancel.

Here is another example used to show that the following very short argument is valid:

S
∴ (S→R)→R

1.	Show (S→R)→R		
2.	S→R	ass cd	<the goal is now to derive the consequent: R>
3.	S	pr	
4.	R	2 3 mp	

We have assumed the antecedent of the conditional on the show line, and we have now succeeded in deriving the consequent of that conditional. So we may use the technique of conditional derivation:

1.	~~Show~~ (S→R)→R	
2.	S→R	ass cd
3.	S	pr
4.	R	2 3 mp cd

EXERCISES

1. For each of the following derivations, determine which lines are correct and which incorrect. (In assessing a line, assume that previous lines are correct.)

a. P → (~Q→R)
~R
∴ P → Q

1.	~~Show~~ P → Q	
2.	P	ass cd
3.	P → (~Q→R)	pr
4.	~Q → R	2 3 mp
5.	~R	pr
6.	~~Q	4 5 mt
7.	Q	6 dn cd

b. P → (Q→~R)
R
∴ P → ~Q

1.	~~Show~~ P → ~Q	
2.	P	ass cd
3.	~Q	1 2 mp
4.	P → (Q→~R)	pr
5.	Q → ~R	2 4 mp
6.	R	pr
7.	~Q	5 6 mt cd

c. $\sim S \to (Q \to R)$
$R \to \sim(Q \to R)$
$\sim P \to R$
$\therefore P \to \sim S$

1. ~~Show~~ $P \to \sim S$

2.	$\sim S$	ass cd
3.	$Q \to R$	2 mp
4.	$\sim\sim(Q \to R)$	3 dn
5.	$R \to \sim(Q \to R)$	pr
6.	$\sim R$	4 5 mt
7.	$\sim P \to R$	pr
8.	$\sim\sim P$	6 7 mt
9.	P	8 dn cd

2. Construct correct derivations for each of the following arguments using conditional derivations.

a. $P \to (Q \to (R \to S))$
$\sim Q \to \sim R$
R
$\therefore P \to S$

b. $Q \to \sim(R \to S)$
$P \to (R \to S)$
$\sim Q \to R$
$\therefore P \to S$

c. $U \to (U \to V)$
$\sim R \to \sim(U \to V)$
$R \to \sim S$
$\therefore U \to \sim S$

3. Symbolize the following arguments using the sentence letters given, and then give derivations to validate them.

a. If Seymour likes papayas he'll have them for tea. He won't have them for tea if we don't have any. If we didn't shop yesterday, we don't have any papayas. So if Seymour likes papayas, we shopped yesterday. (P: Seymour likes papayas; T: Seymour will have papayas for tea; X: We have some papayas; S: We shopped yesterday.)

b. If today is Thursday, then Saturday is two days from now. If the party is on Saturday, then if Saturday is two days from now, then so is the party. I can't go to the party if it's two days from now. The party is on Saturday. So if today is Thursday, I can't go to the party. (T: Today is Thursday; S: Saturday is two days from now; P: The party is two days from now; Y: The party is on Saturday; X: I can go to the party.)

c. If Samantha is at home, she won't cause any trouble. If she isn't at home, she can't be reached by telephone. If she can't be reached by telephone, it's too late to tell her about the party. If she comes to the party she will cause some trouble. So if it's not too late to tell her about the party, she won't come. (S: Samantha is at home' T: Samantha will cause trouble; R: Samantha can be reached by telephone; X: It's too late to tell Samantha about the party; Y: Samantha will come to the party.)

7 INDIRECT DERIVATIONS

There is a third technique for doing derivations, called indirect derivation. It is often used in cases where direct derivation and conditional derivation do not obviously apply. An example is an attempt to show that this inference is valid:

> *Polk was a president*
> *Whitney wasn't a president*
> ∴ *It is not the case that if Polk was a president, so was Whitney*

The conclusion is not a conditional, so a straightforward application of conditional derivation does not seem possible. Nor is it clear how to derive the conclusion using mp, mt, or dn. To validate this argument you might reason as follows:

> We want to show that it is not the case that if Polk was a president, so was Whitney. Well, assume the opposite: assume that it <u>is</u> the case that if Polk was a president then so was Whitney. Then, since we are given that Polk was a president, so was Whitney. But we are given that Whitney wasn't. So we are led to absurd conclusions: Whitney was a president and Whitney was not a president. So the assumption we made, which lead to these inferences, must not be true (given the premises of the argument).

The reasoning is called indirect because in order to show something, you assume the opposite and derive contradictory sentences from it (along with the premises). (Sentences are **contradictory** when one is the negation of the other.) If you succeed in doing this, you have shown that the negation of what you are trying to derive isn't true; it can't be true because it entails contradictory sentences. So what you are trying to derive must itself be true.

The technique of indirect derivation has two parts. First, there is a new kind of assumption: immediately following a show line you may assume the opposite of the sentence on the show line. (The opposite of the sentence is its negation, or its "unnegation" if it is already a negation.) Then when you have two sentences, one of which is the negation of the other, after the last one derived you add the line number of the other and write "id" for "indirect derivation"; then you box and cancel. (Alternatively, you may write a later line with no sentence on it, citing the line numbers of both of the contradictory sentences, write "id'; and then box and cancel.)

Here is a formal derivation corresponding to the above reasoning:

> P
> ~W
> ∴ ~(P→W)

1.	Show ~(P→W)		
2.	P→W	ass id	← *assumption for indirect derivation*
3.	P	pr	
4.	W	2 3 mp	
5.	~W	pr	

Having reached line 5 we may add to it the line number 4 and "id"; then box and cancel:

1.	~~Show~~ ~(P→W)		*Cancel the "Show"*
2.	P→W	ass id	
3.	P	pr	← *Box the lines*
4.	W	2 3 mp	
5.	~W	pr 4 id	← *Write "id"*

The "4 id" at the end of line 5 indicates that the sentence on line 4 contradicts that on the current line, line 5, and thus the assumption (on line 2) which lead to them must be false (given the premises of the argument).

Here is another example of a derivation for this short valid argument:

$U \rightarrow S$
$\sim U \rightarrow S$
$\therefore\ S$

1.	Show S	
2.	$\sim S$	ass id
3.	$U \rightarrow S$	pr
4.	$\sim U$	2 3 mt
5.	$\sim U \rightarrow S$	pr
6.	$\sim\sim U$	2 5 mt

Since we have derived ~U on line 4 and ~~U on line 6, we are in a position to box and cancel:

1.	~~Show~~ S	
2.	$\sim S$	ass id
3.	$U \rightarrow S$	pr
4.	$\sim U$	2 3 mt
5.	$\sim U \rightarrow S$	pr
6.	$\sim\sim U$	2 5 mt 4 id

Notice that we could have used double negation on line 6 to get U, and then this would contradict ~U on line 4. But we needn't do this extra step, since the sentences on lines 4 and 6 already contradict one another.

We could also have postponed the id step to a later line:

1.	~~Show~~ S	
2.	$\sim S$	ass id
3.	$U \rightarrow S$	pr
4.	$\sim U$	2 3 mt
5.	$\sim U \rightarrow S$	pr
6.	$\sim\sim U$	2 5 mt
7.		4 6 id

THINKING UP DERIVATIONS: Suppose that you have an argument and you are not sure whether or not it is valid. You can show it not to be valid by producing what is usually called a "counter-example" -- a logically possible situation in which its premises are all true and its conclusion false. If you get a counter-example, the argument is invalid. Suppose that you aren't able to find a counter-example. That might be because the argument is valid, or it might be because you haven't been lucky enough or clever enough to find a counter-example. So failing to find a counter-example, by itself, shows nothing. However, sometimes when you try to find a counter-example, and you fail, this is because any attempt to make the premises true and conclusion false leads you to assign opposite truth values to some sentence. If this happens to you, your failure can be used as a guide to producing a derivation that validates the argument. Typically, it is a guide to producing an indirect derivation.

Here is an example. You are given the argument:

If Pedro left early then if Taylor stayed, Zack left
Taylor stayed
If Zack left, Pedro didn't leave early
∴ *Pedro didn't leave early*

Symbolized, it is:

$P \to (T \to Z)$
T
$Z \to \sim P$
$\therefore\ \sim P$

Consider an attempt to produce a counter-example. You need a case in which the conclusion, '~P', is false; that is, one in which 'P' is true. So assume that 'P' is true. By the first premise, that makes 'T→Z' be true, and then by the second premise, 'Z' must be true. Then by the third premise, '~P' must be true. Oops! We can't make both 'P' and '~P' true. So there must be no counter-example.

Here is a derivation based on that reasoning. The assumption that 'P' is true is just like an assumption for purposes of indirect derivation:

1.	Show ~P	
2.	P	ass id

Now the next few steps of our attempt to get a counter-example are paralleled by familiar steps in a derivation:

1.	Show ~P	
2.	P	ass id
3.	$P \to (T \to Z)$	pr
4.	$T \to Z$	2 3 mp
5.	T	pr
6.	Z	4 5 mp
7.	$Z \to \sim P$	pr
8.	~P	6 7 mp

The "Oops" in the failed counter-example search parallels the fact that we now have contradictories on lines 2 and 8. So we may box and cancel:

1.	~~Show~~ ~P	
2.	P	ass id
3.	$P \to (T \to Z)$	pr
4.	$T \to Z$	2 3 mp
5.	T	pr
6.	Z	4 5 mp
7.	$Z \to \sim P$	pr
8.	~P	6 7 mp 2 id

This then is an indirect derivation that validates the argument.

EXERCISES

1. For each of the following derivations, determine which lines are correct and which incorrect. (In assessing a line, assume that previous lines are correct.)

a. $R \to (S \to T)$
S
$\sim T$
$\therefore\ \sim R$

1. ~~Show~~ $\sim R$

2.	R	ass id
3.	$R \to (S \to T)$	pr
4.	$S \to T$	2 3 mp
5.	S	pr
6.	T	4 5 mp
7.	$\sim T$	pr 6 id

b. $\sim S \to P$
R
$S \to \sim R$
$\therefore$ P

1. ~~Show~~ P

2.	R	pr
3.	$\sim S \to \sim R$	pr
4.	$\sim S$	2 3 mt
5.	$\sim P$	ass id
6.	$\sim S \to P$	pr
7.	$\sim\sim S$	5 6 mt 5 id

c. $U \to (V \to \sim W)$
$X \to (U \to V)$
$(V \to \sim W) \to X$
$\therefore$ $U \to V$

1. ~~Show~~ $U \to V$

2.	$U \to \sim V$	ass id
3.	$U \to (V \to \sim W)$	pr
4.	$\sim V$	2 3 mt
5.	$X \to (U \to V)$	pr
6.	$\sim X$	2 5 mt
7.	$\sim(U \to V)$	5 6 mt 2 id

2. Construct correct derivations for each of the following arguments using indirect derivations.

a. $\sim Q \to R$
$S \to \sim R$
$\sim S \to Q$
$\therefore$ Q

b. $(P \to Q) \to R$
$S \to (P \to Q)$
$\sim S \to R$
$\therefore$ R

c. $\sim P \to (R \to S)$
$(R \to S) \to T$
$\sim T$
$Q \to (R \to S)$
$\therefore$ $\sim(P \to Q)$

8 SUBDERIVATIONS

Reasoning can be intricate. One way in which this happens is when derivations occur within derivations. Consider the following argument in symbolic form:

$\sim(R\to Q) \to P$
$P \to (\sim Q\to Q)$
$\sim Q$
$\therefore\ \sim R$

1. Show $\sim R$

It is not clear how to directly derive ~R using our rules mp, mt, and dn. A conditional derivation is not applicable because ~R is not a conditional. One could assume R for purposes of doing an indirect derivation, but it is not clear how to proceed from there. An alternative approach is to use a derivation within the main derivation. Here is one way to proceed:

> Try to derive the negation of '$\sim Q\to Q$' from the third premise, and then use modus tollens on the second, and then on the first, premise to get $R\to Q$, which then leads to the desired conclusion using mt and the third premise.

The first part of this strategy -- deriving '$\sim(\sim Q\to Q)$' from the third premise -- requires a derivation of its own. You can do this as a conditional derivation, leaving you with:

1.	Show $\sim R$	
2.	~~Show~~ $\sim(\sim Q\to Q)$	
3.	$\sim Q\to Q$	ass id
4.	$\sim Q$	pr
5.	Q	3 4 mp 4 id

You now may use line 2 just as you would use any other line; once the "Show" is cancelled, the line is no longer something you are trying to derive, it is something you have derived. (A cancelled 'show' means "shown".) You proceed:

1.	Show $\sim R$	
2.	~~Show~~ $\sim(\sim Q\to Q)$	
3.	$\sim Q\to Q$	ass id
4.	$\sim Q$	pr
5.	Q	3 4 mp 4 id
6.	$P \to (\sim Q\to Q)$	pr
7.	$\sim P$	2 6 mt
8.	$\sim(R\to Q) \to P$	pr
9.	$\sim\sim(R\to Q)$	7 8 mt
10.	$R\to Q$	9 dn
11.	$\sim Q$	pr
12.	$\sim R$	10 11 mt

This is a complete direct derivation, so you may box and cancel:

1. ~~Show~~ ~(~Q→Q)
2. ~~Show~~ ~(~Q→Q)
3. ~Q→Q ass id
4. ~Q pr
5. Q 3 4 mp 4 id
6. P → (~Q→Q) pr
7. ~P 2 6 mt
8. ~(R→Q) → P pr
9. ~~(R→Q) 7 8 mt
10. R→Q 9 dn
11. ~Q pr
12. ~R 10 11 mt dd

The subderivation was just a way to derive '~(~Q→Q)'.

Here is another example:

(P→~Q) → (R →S)
Q→~P
T→R
∴ T→S

We begin the derivation by stating what is to be shown:

1. Show T→S

Generally, the easiest way to derive a conditional is to use conditional derivation. So we write this:

1. Show T→S
2. T ass cd

with the goal of deriving S, thereby completing the conditional derivation. At this point it is unclear what to do next. However, we notice that S occurs only once in the premises -- in the first premise. And if we could derive the antecedent of that premise, we could use mp to infer R→S, which would get us close to completing the derivation. So we try to derive the antecedent of the first premise:

1. Show T→S
2. T ass cd
3. Show P→~Q

Line 3 is a conditional, so we try to use a conditional derivation:

1. Show T→S
2. T ass cd
3. Show P→~Q
4. P ass cd

Our immediate goal is now to derive ~Q. We can do that by appealing to the double negation of P along with the second premise:

1. Show T→S
2. T ass cd
3. Show P→~Q
4. P ass cd
5. ~~P 4 dn
6. Q→~P pr
7. ~Q 5 6 mt

This completes the conditional subderivation, so we box and cancel:

1.	Show T→S	
2.	T	ass cd
3.	~~Show~~ P→~Q	
4.	P	ass cd
5.	~~P	4 dn
6.	Q→~P	pr
7.	~Q	5 6 mt

We may now do the rest of the derivation:

1.	Show T→S	
2.	T	ass cd
3.	~~Show~~ P→~Q	
4.	P	ass cd
5.	~~P	4 dn
6.	Q→~P	pr
7.	~Q	5 6 mt cd
8.	(P→~Q) → (R→S)	pr
9.	R→S	3 8 mp
10.	T→R	pr
11.	R	2 10 mp
12.	S	9 11 mp

Line 12 completes the main conditional derivation, so we box and cancel, and we are done:

1.	~~Show~~ T→S	
2.	T	ass cd
3.	~~Show~~ P→~Q	
4.	P	ass cd
5.	~~P	4 dn
6.	Q→~P	pr
7.	~Q	5 6 mt cd
8.	(P→~Q) → (R→S)	pr
9.	R→S	3 8 mp
10.	T→R	pr
11.	R	2 10 mp
12.	S	9 11 mp cd

Now that we have derivations within derivations, the availability of previous lines for the purpose of applying rules can change from line to line; a line that is not available at one point can become available later, and one that is available may become unavailable. Examples:

> At line 4 above, line 3 is not available, because from the point of view of line 4, line 3 is an uncancelled show line. But the 'show' on line 3 is cancelled at line 7, so from the point of view of line 8, line 3 *is* available.
>
> On the other hand, at line 6, line 5 is available; but line 5 is no longer available at line 8, because it has been boxed at line 7.

We can now state explicitly what may appear in a derivation which may contain subderivations:

DERIVATIONS

A **derivation** from a set of sentences P consists of a sequence of lines that is built up in order, step by step, where each step is in accordance with these provisions:

- **Show line:** A show line consists of the word "Show" followed by a symbolic sentence. A show line may be introduced at any step. Show lines are not given a justification.
- **Premise**: At any step, any symbolic sentence from the set P may be introduced, justified with the notation "pr".
- **Rule:** At any step, a line may be introduced if it follows by a rule from sentences on previous available lines; it is justified by citing the numbers of those previous lines and the name of the rule.
- **Direct derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the sentence on the closest previous uncancelled show line, one may, as the next step, write "dd" following the justification for that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for conditional derivation**: When a show line with a conditional sentence is introduced, as the next step one may introduce an immediately following line with the antecedent of the conditional on it; the justification is "ass cd".
- **Conditional derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the consequent of the conditional sentence on the closest previous uncancelled show line, one may, as the next step, write "cd" at the end of that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for indirect derivation**: When a show line is introduced, as the next step one may introduce an immediately following line with the [un]negation of the sentence on the show line; the justification is "ass id".
- **Indirect derivation:** When a sentence is introduced on a line which is not a show line, if there is a previous available line containing the [un]negation of that sentence, and if there is no uncancelled show line between the two sentences, as the next step you may write the line number of the first sentence followed by "id" at the end of the line with the second sentence. Then you cancel the closest previous "show", and box all sentences below that show line, including the current line.

Except for steps that involve boxing and canceling, every step introduces a line. When writing out a derivation, every line that is introduced is written directly below previously introduced lines.

Optional variant: When boxing and canceling with direct or conditional derivation, the "dd" or "cd" justification may be written on a later line which contains no sentence at all, and which is followed by the number of the line that satisfies the conditions for direct or conditional derivation. With indirect derivation, the "id" justification may be written on a later line which contains no sentence at all, and which is followed by the numbers of the two lines containing contradictory sentences. In both cases, the lines cited must be available from the later line.

EXERCISES

1. For each of the following derivations, determine which lines are correct and which incorrect. (In assessing a line, assume that previous lines are correct.)

Tip: When a box occurs in a correctly formed derivation, it is put there by the command ('dd', or 'cd' or 'id') that appears at the end of the last line within the box. When a "show" is cancelled, it is cancelled by the same command that puts the box immediately below the "show'.

a. $P \to (Q \to R)$
$\sim Q \to S$
$\therefore P \to (\sim S \to R)$

1.	~~Show~~ $P \to (\sim S \to R)$	
2.	P	ass cd
3.	~~Show~~ $\sim S \to R$	
4.	$\sim S$	ass cd
5.	$\sim Q \to S$	pr
6.	$\sim\sim Q$	4 5 mt
7.	Q	6 dn
8.	$P \to (Q \to R)$	pr
9.	$Q \to R$	2 8 mp
10.	R	7 9 mp cd
11.		3 cd

b. $R \to Q$

$Q \to P$

$\therefore$ $R \to P$

1.	~~Show~~ $R \to P$	
2.	R	ass cd
3.	$R \to Q$	pr
4.	Q	2 3 mp
5.	~~Show~~ P	
6.	$\sim P$	ass id
7.	$Q \to P$	pr
8.	$\sim Q$	6 7 mt 4 id
9.		5 cd

error on this line!

c. $P \to Q$

$(R \to Q) \to S$

$(U \to S) \to \sim P$

$\therefore$ $\sim P$

1.	~~Show~~ $\sim P$	
2.	P	ass id
3.	~~Show~~ $R \to Q$	
4.	R	ass cd
5.	$P \to Q$	pr
6.	Q	2 6 mp cd
7.	~~Show~~ $U \to S$	
8.	U	ass cd
9.	$(R \to Q) \to S$	pr
10.	S	3 9 mp cd
11.	$(U \to S) \to \sim P$	pr
12.	$\sim P$	7 11 mp
13.	P	2 r 12 id

2. Construct correct derivations for each of the following arguments.

a. $P \rightarrow (Q \rightarrow R)$
$S \rightarrow Q$
$\therefore\ S \rightarrow (P \rightarrow R)$

b. $(P \rightarrow Q) \rightarrow Q$
$P \rightarrow R$
$Q \rightarrow \sim Q$
$\therefore\ \sim(R \rightarrow Q)$

c. $(U \rightarrow V) \rightarrow (W \rightarrow X)$
$U \rightarrow Z$
$\sim V \rightarrow \sim Z$
$X \rightarrow Z$
$\therefore\ W \rightarrow Z$

9 SHORTCUTS

Writing long derivations can be tedious. Here are two shortcuts.

Citing premises without rewriting them

Every time we have wanted to appeal to a premise when using a rule of inference, we have written the premise into the derivation, justifying it by 'pr', and then when we use the rule we cite the number of the line where the premise was written. We can skip writing down the premise entirely if the justification of the rule identifies the premise by name. For example, instead of having:

8.	$Q \to R$		
9.	Q	pr	(where 'Q' is the third premise)
10.	R	8 9 mp	

we just write:

8.	$Q \to R$	
9.	R	8 pr3 mp

This is equivalent to just assuming that the premises all come with line numbers: pr1, pr2, pr3, . . ., and citing those line numbers when we use a rule of inference. (Our directions above for constructing derivations already include this option.)

For example, here is a derivation we gave earlier:

Premises:		$P\to(W\to T)$
		$\sim P\to T$
		$\sim T$
Conclusion:	$\therefore$	$\sim W$

1.	~~Show~~ ~W	
2.	~T	pr
3.	$\sim P\to T$	pr
4.	~~P	2 3 mt
5.	P	4 dn
6.	$P\to(W\to T)$	pr
7.	$W\to T$	5 6 mp
8.	~T	pr
9.	~W	7 8 mt dd

Using the shortcut, we can essentially skip lines 2, 3, 6, 8, to get this shortened derivation which has analogues of lines 1, 4, 5, 7, 9 in the original derivation:

1.	~~Show~~ ~W	
2.	~~P	pr2 pr3 mt
3.	P	2 dn
4.	$W\to T$	3 pr1 mp
5.	~W	4 pr3 mt dd

Explicitly, the technique is:

Citing Premises in Rules

When citing a premise in applying a rule of inference, use 'pr1', 'pr2' 'pr3', . . . to identify the first, second, third, . . . premises.

Mixed derivations: Our derivation rules are already formulated in a way that lets you use any one of dd, cd, id, in cases in which you except to use another of them. For example, suppose you are trying to show 'P→Q' by cd, assuming 'P'. But you derive 'P→Q' instead of 'Q', Then you can use dd to box and cancel. The fact that you have assumed 'P' for purposes of producing a conditional derivation does not interfere with this use of dd. Given these lines:

1.	Show □→○	
2.	□	ass cd
3.	……..	
7.	……..	
8.	□→○	

you can box and cancel:

1.	~~Show~~ □→○	
2.	□	ass cd
3.	……..	
7.	……..	
8.	□→○	dd

This is a "mixed" derivation: an assumption is made for constructing a conditional derivation, and then 'dd' is used instead of 'cd' to complete it. That's OK because this is just a shortcut. Whenever you are in the position described above, you could instead add a step to the end of the derivation and then conclude the derivation as a conditional derivation. Just add step 9:

1.	Show □→○	
2.	□	ass cd
3.	……..	
4.	……..	
7.	……..	
8.	□→○	
9.	○	2 8 mp

and then box and cancel using cd:

1.	~~Show~~ □→○	
2.	□	ass cd
3.	……..	
4.	……..	
7.	……..	
8.	□→○	
9.	○	2 8 mp cd

So using dd at the end of line 8 is merely a way to save a step.

Similarly, if you are trying to do an indirect derivation and you end up deriving the sentence on the show line, you may use dd. That is, if you have:

1.	Show □	
2.	~□	ass id
3.	……..	
4.	……..	
7.	……..	
8.	□	

you may box and cancel with dd:

1. ~~Show~~ □
2. ~□ ass cd
3. ……..
4. ……..

7. ……..
8. □ dd

Here the shortcut is obvious; you are already in a position to use id since you already have the contradictory sentences that you need; you can instead cite the line number of the other contradictory and use id:

1. ~~Show~~ □
2. ~□ ass cd
3. ……..
4. ……..

7. ……..
8. □ 2 id

Similarly you can use cd when you are set up for a direct derivation of a conditional, or when you are trying to derive a conditional using id; and you can use id when you have derived contradictories even if you are set up for a direct or conditional derivation.

In allowing for mixed derivations, we are not actually changing anything. Our rules already allow for them. So this is a summary of things we can already do with the rules as stated:

Mixed derivations
You may use dd, cd, and id to complete a derivation by boxing and canceling whenever they apply, whether or not an assumption has been made, and regardless of the type of assumption if any.

EXERCISES

1. Each of the following derivations is a mixed derivation. In each case produce another derivation which is not mixed.

a. $P \rightarrow R$
$Q \rightarrow \sim R$
$\sim Q \rightarrow Q$
$\therefore$ $P \rightarrow Q$

1. ~~Show~~ $P \rightarrow Q$

2.	P	ass cd
3.	$P \rightarrow R$	pr
4.	R	2 3 mp
5.	$\sim\sim R$	4 dn
6.	$Q \rightarrow \sim R$	pr
7.	$\sim Q$	5 6 mt
8.	$\sim Q \rightarrow Q$	pr
9.	Q	7 8 mp 7 id

b. $Q \rightarrow U$
$Q \rightarrow \sim U$
$R \rightarrow Q$
R
$\therefore$ P

1. ~~Show~~ P

2.	R	pr
3.	$R \rightarrow Q$	pr
4.	Q	2 3 mp
5.	$Q \rightarrow U$	pr
6.	U	4 5 mp
7.	$Q \rightarrow \sim U$	pr
8.	$\sim U$	4 7 mp 6 id

c. $U \rightarrow (V \rightarrow W)$
$X \rightarrow U$
$\sim X \rightarrow W$
$\therefore$ $V \rightarrow W$

1. ~~Show~~ $V \rightarrow W$

2.	$\sim(V \rightarrow W)$	ass id
3.	$U \rightarrow (V \rightarrow W)$	pr
4.	$\sim U$	2 3 mt
5.	$X \rightarrow U$	pr
6.	$\sim X$	4 5 mt
7.	$\sim X \rightarrow W$	pr
8.	W	6 7 mp cd

2. Do the derivations from 1a-c above using premise line numbers instead of rule pr.

3. Earlier we stipulated that a conditional sentence is false when its antecedent is true and its consequent false, and true in all other cases. This was not arbitrary. Given the rules and derivation procedures that we have adopted, these choices are forced on us. For, using our rules and procedures, we can produce derivations to show each of the following arguments to be valid:

P	P	~P	~P
Q	~Q	Q	~Q
∴ P→Q	∴ ~(P→Q)	∴ P→Q	∴ P→Q

The first of these tells us that when the antecedent and consequent of a conditional are both true, so is the conditional. The second tells us that when the antecedent of a conditional is true and the consequent false, the conditional is false. The third and fourth tell us that in either case in which the antecedent is false, the conditional is true.

Produce short derivations for each of those four arguments.

10 STRATEGY HINTS FOR DERIVATIONS

Some strategies are generally useful in thinking up how to construct derivations. They do not always work, but they are often the best way to begin the search for a successful derivation.

1. Try to reason out the argument for yourself. If you can do that, then write down the steps you went through in your own reasoning. These steps will often be an outline of a good derivation. This idea of reasoning things out and then turning the steps into a derivation has been illustrated above when introducing direct, conditional, and indirect derivations. This is by far the best approach to thinking up a derivation.

2. Begin with a sketch of an outline of a derivation, and then fill in the details. For example, in thinking up a derivation for this argument:

P → (Q→R)
Q
∴ P → R

you might begin by saying: "I'll do a conditional derivation: I'll assume P and then use this together with the premises to derive R. Then I'll box and cancel by cd." This outline gives you this much of a derivation:

1.	Show P→ R		
2.	P	ass cd	
	:::::		
	:::::		
	:::::		
13.	R	cd	. . . . then box and cancel

All you need to do then is to fill in lines 3 to 12. (I'm just guessing that it will take exactly eight more steps to finish the derivation. If it takes more or less, change the '13' to the appropriate number.)

3. Write down obvious consequences: Write down obvious consequences of premises or of sentences that have already been derived. If you write down the simple consequences of things that you already have, then you have lots of resources right in front of you. Of course, you can do this in your head instead of on paper, and sometimes that is sufficient. But if things are not completely clear to you, writing down obvious consequences can be useful.

Suppose you are trying to construct a derivation to validate this argument:

$P \rightarrow (Q \rightarrow R)$
$S \rightarrow Q$
$Q \rightarrow P$
S
$\therefore R$

Let's say that you have written down the show line:

1. Show R

but you are momentarily stuck. Write down some obvious consequences of the premises:

2.	Q	pr4 pr2 mp
3.	P	pr3 2 mp
4.	$Q \rightarrow R$	pr1 3 mp

At this point, if you look over what is available, it will be obvious that you can get the desired conclusion in one additional step:

5.	R	2 4 mp

4. If you are trying to derive a conditional, use conditional derivation. This is almost always the easiest way to derive a conditional. This has been amply illustrated above.

5. If you have a conditional, try to derive its antecedent (and then use modus ponens), or try to derive the negation of its consequent (and then use modus tollens). Here is an illustration:

$P \rightarrow (Q \rightarrow R)$
$S \rightarrow Q$
$\sim P \rightarrow \sim S$
S
$\therefore R$

The only premise in which R occurs is the first one, which is a conditional. You could apply modus ponens to this conditional if you could prove P. Our strategy rule suggests that you try to derive P. That is not difficult to do:

1.	Show R	
2.	~~S	pr4 dn
3.	~~P	2 pr3 mt
4.	P	3 dn

This immediately gives us:

5.	$Q \rightarrow R$	4 pr1 mp

It is then easy to complete the derivation:

6.	Q	pr2 pr4 mp
7.	R	5 6 mp

and you are ready to box and cancel.

6. Try indirect derivation. When you reach a place where none of the other strategies clearly apply, assume the negation of what you are trying to derive and try to derive a contradiction. This too has been amply illustrated above.

7. When doing an indirect derivation, try to derive the negation of a premise or the negation of something that you already have derived. This is especially useful if you already have the negation of a conditional, for you can try to derive the conditional by using conditional derivation, and then you will have both the conditional and its negation, and you can box and cancel with id. An example. You have:

R
$(Q \to S) \to {\sim}R$
$\therefore\ {\sim}S$

Now begin a derivation, and follow strategy rule 1: write out obvious consequences of what you have:

1.	Show $\sim S$	
2.	S	ass id
3.	$\sim\sim R$	pr1 dn
4.	$\sim(Q \to S)$	3 pr2 mt

The moves so far are pretty straightforward. But it may not be clear what to do next. Strategy rule 7 suggests that you try to derive the conditional '$Q \to S$', in order to contradict '$\sim(Q \to S)$'. You do this with a conditional subderivation:

5.	~~Show~~ $Q \to S$	
6.	Q	ass cd
7.	S	2 r cd

(*this step is obvious once you notice it*)

Having derived the conditional on line 5 and its negation on line 4 the indirect derivation is just about complete. The complete derivation is:

1.	~~Show~~ $\sim S$	
2.	S	ass id
3.	$\sim\sim R$	pr1 dn
4.	$\sim(Q \to S)$	3 pr2 mt
5.	~~Show~~ $Q \to S$	
6.	Q	ass cd
7.	S	2 r cd
8.		4 5 id

EXERCISES

Produce correct derivations to validate these arguments.

1. S
   $(R \to S) \to W$
   $\therefore\ W$

2. $P \to (S \to R)$
   $P \to (W \to S)$
   $W \to P$
   $\therefore\ W \to R$

3. $(P \to Q) \to S$
   $S \to T$
   ${\sim}T \to Q$
   $\therefore\ T$

11 THEOREMS

A truth of logic (a sentence that is logically true) is a sentence that is true in any logically possible situation. It must be true no matter what. Because of this, a truth of logic is like the conclusion of a valid argument which has no premises. If such an argument is valid, it does not have all true premises and a false conclusion in any logically possible situation. When there aren't any premises, this is equivalent to saying that it does not have a false conclusion in any logically possible situation. That is, its conclusion is true in every logically possible situation. It is a truth of logic.

Since derivations show arguments valid, if a derivation is used to show an argument with no premises to be valid, that amounts to showing that the conclusion is logically true. There is a special word, 'theorem', that refers to any sentence shown by a technique like a derivation when no premises at all are used. So the topic of this section is Theorems.

It is customary to indicate a theorem by placing a "therefore" sign in front of it, as if it were an argument with its premises missing. So writing "∴□" indicates that □ is a theorem.

Here is a derivation to show that 'P→P' is a theorem:

∴ P→P

1.	~~Show~~ P→P	
2.	P	ass cd
3.		2 cd

In simple cases, theorems are obviously trivial statements. Even when they are complex, they are still trivial in the sense that they say nothing beyond what is logically true. In this book we will list several theorems, giving them the names "T1", "T2", and so on. Here are some given in increasing degrees of complexity. Some of them have common names; these are indicated to the right.

T1 **P→P** <just proved>

T2 **Q → (P→Q)**

1.	~~Show~~ Q → (P→Q)	
2.	Q	ass cd
3.	~~Show~~ P→Q	
4.	P	ass cd
5.	Q	2 r cd
6.		3 cd

T3 **P → ((P→Q) → Q)**

1.	~~Show~~ P → ((P→Q) → Q)	
2.	P	ass cd
3.	~~Show~~ (P→Q) → Q	
4.	P→Q	ass cd
5.	Q	2 4 mp cd
6.		3 cd

T4 **(P→Q) → ((Q→R) → (P→R))** "Syllogism"

T5 **(Q→R) → ((P→Q) → (P→R))** "Syllogism"

T6 $\mathbf{(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))}$ "Distribution of $\to$ over $\to$"

1.	~~Show~~ $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	
2.	$P \to (Q \to R)$	ass cd
3.	~~Show~~ $(P \to Q) \to (P \to R)$	
4.	$P \to Q$	ass cd
5.	~~Show~~ $P \to R$	
6.	P	ass cd
7.	Q	4 6 mp
8.	$Q \to R$	2 6 mp
9.	R	7 8 mp cd
10.		5 cd
11.		3 cd

T7	$\mathbf{((P \to Q) \to (P \to R)) \to (P \to (Q \to R))}$	"Distribution of $\to$ over $\to$"
T8	$\mathbf{(P \to (Q \to R)) \to (Q \to (P \to R))}$	Commutation
T9	$\mathbf{(P \to (P \to Q)) \to (P \to Q)}$	
T10	$\mathbf{((P \to Q) \to Q) \to ((Q \to P) \to P)}$	
T11	$\mathbf{\sim\sim P \to P}$	Double negation
T12	$\mathbf{P \to \sim\sim P}$	Double negation
T13	$\mathbf{(P \to Q) \to (\sim Q \to \sim P)}$	Transposition
T14	$\mathbf{(P \to \sim Q) \to (Q \to \sim P)}$	Transposition
T15	$\mathbf{(\sim P \to Q) \to (\sim Q \to P)}$	Transposition
T16	$\mathbf{(\sim P \to \sim Q) \to (Q \to P)}$	Transposition
T17	$\mathbf{P \to (\sim P \to Q)}$	
T18	$\mathbf{\sim P \to (P \to Q)}$	
T19	$\mathbf{(\sim P \to P) \to P}$	Reductio ad absurdum
T20	$\mathbf{(P \to \sim P) \to \sim P}$	Reductio ad absurdum
T21	$\mathbf{\sim(P \to Q) \to P}$	
T22	$\mathbf{\sim(P \to Q) \to \sim Q}$	
T23	$\mathbf{((P \to Q) \to P) \to P}$	Peirce's law

EXERCISES

1. Prove theorems T8, T11, T13, T17, T19, T21, T22.

12 USING PREVIOUSLY PROVED THEOREMS IN DERIVATIONS

One major use of theorems is to avoid repeating derivations that you have done earlier. For example, suppose you have previously derived 'P → (P → (Q→P))' from no premises. Now you are doing a new derivation where you have derived 'R', and you want to derive 'S→R'. By following out the reasoning from earlier, but using 'R' instead of 'P', you could derive this: 'R → (R → (S→R))', and then use modus ponens twice to get 'S → R'. But why write out the new derivation of 'R → (R → (S→R))' when it may be lengthy, and it just involves repeating moves that you made earlier? It would be nice to have a way to "reuse" old derivations for new purposes when the reasoning is the same. One way to do this is with this new procedure that we adopt hereafter:

> **Theorems**: Any instance of any previously derived theorem may be entered on any line of a derivation. As justification, write the name of the theorem. (E.g. 'T13'.)

An instance of a theorem is what you get by considering the theorem as a pattern, and filling in the pattern uniformly with sentences. For example, T1 is ∴P→P. This validates the pattern '□→□'. Anything got by filling in that pattern, putting the same sentence in for each occurrence of □, can be written on any line of a derivation. For example, you can write:

21. (S→W) → (S→W) T1

More complicated theorems are more useful. For example, T4 is:

(P→Q) → ((Q→R) → (P→R))

This gives us the pattern:

(□→○) → ((○→△) → (□→△))

Putting ~R in for □, (U→V) for ○, and W for △ we have:

(~R→(U→V)) → (((U→V)→W) → (~R→W))

Here is an example of a use of this theorem. Suppose you wish to do a derivation for this argument:

R→~S
~S→~T
∴ R→~T

A very short derivation could be given using theorem 4:

1. ~~Show~~ R→~T

2.	(R→~S) → ((~S→~T) → (R→~T))	T4
3.	(~S→~T) → (R→~T)	2 pr1 mp
4.	R→~T	3 pr2 mp dd

EXERCISES

1. Produce short derivations for these arguments using instances of the theorems listed above.

a. X → ~(Y → Z)
∴ (Y → Z) → ~X

b. R → (~P→S)
R → ~P
∴ R → S

c. $\sim(R \rightarrow (S \rightarrow T))$
$R \rightarrow P$
$P \rightarrow (Q \rightarrow (S \rightarrow T))$
$\therefore \sim Q$

d. $Q \rightarrow R$
$R \rightarrow S$
$\therefore Q \rightarrow S$