Chapter Two
Sentential Logic with 'and', 'or', if-and-only-if'

1 SYMBOLIC NOTATION

In this chapter we expand our formal notation by adding three two-place connectives, corresponding roughly to the English words 'and', 'or' and 'if and only if':

∧	and
∨	or
↔	if and only if

Conjunction: The first of these, '∧', is the conjunction sign; it has the same logical import as 'and'. It goes between two sentences to form a complex sentence which is true if both of the parts (called '**conjuncts**') are true, and is otherwise false:

□	○	(□∧○)
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

Disjunction: The disjunction sign, '∨', makes a sentence that is true in every case except when its parts (its **disjuncts**) are both false. This corresponds to one use (the "inclusive" use) of 'or' in English:

□	○	(□ ∨ ○)
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Biconditional: The biconditional sign, '↔', states that both of the parts making it up (its **constituents**) are the same in truth value. It works like this:

□	○	(□ ↔ ○)
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Each of these new connectives behaves syntactically just like the conditional sign, '→': you make a bigger sentence out of two sentences plus a pair of parentheses:

(□∧○)
(□ ∨ ○)
(□ ↔ ○)

Our expanded definition of a sentence in official notation is now:

Chapter Two SYMBOLIC SENTENCES

- Any capital letter between 'P' and 'Z' is a symbolic sentence.
- If □ is a symbolic sentence, so is ~□
- If □ and ○ are symbolic sentences, so are (□→○), (□∧○), (□ ∨ ○), and (□ ↔ ○).

Nothing is a symbolic sentence for purposes of chapter 2 unless it can be generated by the clauses given above.

As before, we allow ourselves informally to omit the outer parentheses when the sentence occurs alone on a line. It is also customary (and convenient) to omit parentheses around conjunctions or disjunctions when they are combined with a conditional or biconditional sign. The sentence:

$$P\land Q \rightarrow R$$

is to be considered to be an informally worded conditional whose antecedent is a conjunction:

$$(P\land Q) \rightarrow R$$

If we want to make a conjunction whose second conjunct is a conditional, we must use parentheses around the parts of the conditional:

$$P \land (Q\rightarrow R)$$

Likewise, this sentence:

$$P \leftrightarrow Q\lor R$$

is an informally written biconditional whose second constituent is a disjunction:

$$P \leftrightarrow (Q\lor R)$$

If we wish to write a disjunction whose first disjunct is a biconditional, we need to use parentheses around the biconditional:

$$(P\leftrightarrow Q) \lor R.$$

Finally, we may use three conjunction signs or disjunction signs (but not a mix of conjunctions with disjunctions) as abbreviations for what you get by restoring the parentheses by grouping the left two parts together, so that: '$P \land Q \land R$' is an abbreviation for '$(P\land Q) \land R$'.

Informal Conventions

Outermost parentheses may be omitted.

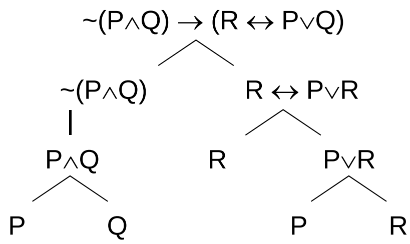
Conjunction signs or disjunction signs may be used with conditional signs or biconditional signs with the understanding that this is short for a conditional or biconditional which has a conjunction or disjunction as a part. For example:

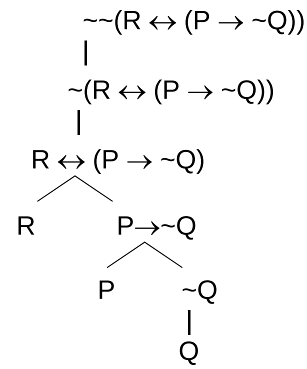
$P\lor Q \rightarrow R$	is informal notation for	$(P\lor Q) \rightarrow R$
$P \leftrightarrow Q\land R$	is informal notation for	$P \leftrightarrow (Q\land R)$

Repeated conjuncts or disjuncts without parentheses are short for the result of putting parentheses around the part to the left of the last conjunction or disjunction sign. For example:

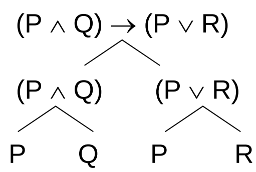
$P \lor Q \lor R$	is informal notation for	$(P\lor Q) \lor R$
$P \land Q \land R$	is informal notation for	$(P\land Q) \land R$

Sentences with the new connectives may be parsed as we did in the previous chapter:

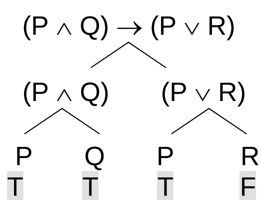




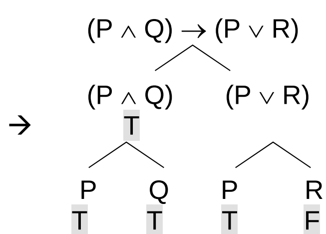
Determining Truth Values Using such parsings, there is a mechanical way to determine whether any given sentence is true or false if you know the truth values of the sentence letters making it up. First, make a parse tree as above by taking the sentences on any given line and writing their immediate parts below them. A parse tree for '(P∧Q) → (P∨R)' is:



Then write the truth values of the sentence letters below them. For example, if P and Q are both true but R false, you would have:



Then go up the parse tree, placing a truth value under the major connective of each sentence based on the truth values of its parts given below. For example, the truth value under '(P ∧ Q)' would be 'T' because it is a conjunction, and both of its parts are T:



Filling in the remaining parts gives you a truth value for the whole sentence at the top:

$((P \wedge Q) \rightarrow (P \vee R))$
T

$(P \wedge Q)$ $(P \vee R)$
T T

P Q P R
T T T F

Sometimes not all of the parse tree needs to be filled out; this happens when partial information below a sentence is sufficient to decide its truth value. In the example just given it is not necessary to figure out the truth value of '$(P \wedge Q)$', since the conditional on the top line is determined to be true based on the information that '$(P \vee R)$' is true. So the following parse tree is sufficient to show that the main sentence is true if the sentence letters have the indicated truth values:

$(P \wedge Q) \rightarrow (P \vee R)$
T

$(P \wedge Q)$ $(P \vee R)$
T

P Q P R
T F

EXERCISES

1. For each of the following state whether it is a sentence in official notation, or a sentence in informal notation, or not a sentence at all. If it is a sentence, parse it as indicated above.

a. $P \leftrightarrow Q \rightarrow R$
b. $\sim Q \leftrightarrow \sim R$
c. $\sim(Q \leftrightarrow R)$
d. $P \wedge Q \vee R$
e. $(P \rightarrow Q) \vee (R \rightarrow \sim Q)$
f. $P \leftrightarrow (Q \wedge R) \rightarrow Q$
g. $P \wedge Q \rightarrow (Q \rightarrow R \vee Q)$
h. $P \leftrightarrow (P \leftrightarrow Q \wedge R)$
i. $P \vee (Q \rightarrow P)$

2. If 'P' and 'Q' are both true and 'R' is false, what are the truth values of the official or informal sentences in 1? (Use the parses that you give in 1 to guide the determination of truth values.)

2 ENGLISH EQUIVALENTS OF THE CONNECTIVES

Conjunctions: The word 'and' is equivalent to the symbol '∧'. There are other locutions of English that may also be equivalent to '∧', although they are sometimes used to communicate something additional. For example:

The book is short, ***and*** *it is interesting*
The book is short, ***but*** *it is interesting*
The book is short, ***although*** *it is interesting*
The book is short, ***even though*** *it is interesting*
*The book is short****;*** *it is interesting*

Some of these sentences suggest that if a book is short, you probably won't find it interesting. But all that they literally say is that it is both short and interesting. If it isn't short, what you have said is false, and if it isn't interesting then what you have said is false, but if it is both short and interesting, what you have said is true, even if possibly misleading.

Conjunctions: □ ∧ ○
□ *and* ○
both □ *and* ○
□ *but* ○
□ *although* ○
although □**,** ○
□ *even though* ○
even though □, ○
□;○

In certain cases, use of a relative pronoun is logically equivalent to a use of '∧': the sentence '*Maria, who was late, greeted the vice-counsel*' is equivalent to '*Maria was late* ∧ *Maria greeted the vice-counsel*'.

Disjunctions: The English word '*or*' can be taken in two ways: inclusively or exclusively. If you are asked to contribute food or money, you will probably take this as saying that you may contribute either or both; the invitation is inclusive. But if a menu says that you may have soup or salad the normal interpretation is that you may have either, but not both; the offer is exclusive. The difference in logical import appears in the first row here:

□	○	(□ inclusive-or ○)	(□ exclusive-or ○)
T	T	**T**	**F**
T	F	**T**	**T**
F	T	**T**	**T**
F	F	**F**	**F**

If the English '*or*' can be read either inclusively or exclusively, we will need to have a convention for how to interpret it when it is used in exercises. Our convention will be that '*or*' is always meant inclusively when it is used in problems and examples in this text. That is, it coincides in logical import with our disjunction sign '∨'.

A common synonym of '*or*' is '*unless*'. The sentence '*Wilma will leave unless there is food*' is false if there is no food but Wilma doesn’t leave; otherwise it is true, just like '*or*' when read inclusively.

Disjunctions: □ ∨ ○
□ *or* ○
either □ *or* ○
□ *unless* ○

Biconditionals: We will see below that a biconditional sign is equivalent to two conditionals made from its constituents. The sentence '(□ ↔ ○)' is equivalent to:

$(\square \to \bigcirc) \land (\bigcirc \to \square)$

This can be read in English as '○ *if* □, *and* ○ *only if* □'; thus it is often pronounced '*if and only if*'. The English phrase '*just in case*' or '*exactly in case*' are sometimes used to state the equivalence of two claims; the biconditional can be used to symbolize them:

The game will be called off just in case it rains:	$Q \leftrightarrow R$
The game will be played exactly in case it is sunny:	$P \leftrightarrow S$

Biconditionals: $\square \leftrightarrow \bigcirc$
□ if and only if ○
□ *exactly on condition that* ○
□ *just in case* ○

EXERCISES

1. For each of the following sentences say which symbolic sentence it is equivalent to.

a. *It will rain, but the game will be played anyway.*
$R \land P$
$R \to P$
$R \leftrightarrow P$

b. *Willa drove or got a ride*
$W \lor R$
$W \leftrightarrow R$

c. *Robert, who didn't get a ride, was tardy*
$\sim R \to T$
$\sim R \land T$

d. *It rained; the sell-a-thon was called off*
$R \leftrightarrow S$
$R \land S$

e. *The quilting bee will be called off just in case it rains*
$Q \land R$
$Q \leftrightarrow R$
$Q \to R$
$R \to Q$

2. Symbolize each of the following using this translation scheme:

S	*Sally will walk*
V	*Veronica will give Sally a ride*
R	*It will rain*
Q	*Barbara will come with Quincy*
T	*Barbara will come with Tom*

a. *Sally will walk or Veronica will give her a ride.*
b. *Exactly on condition that it rains will Sally walk*
c. *Although it will rain, Sally will walk*
d. Barbara *will come with Quincy or Tom*
e. Barbara *will come with Quincy; Sally will walk*

3. What are the truth values of the sentences in 2 when all of the simple sentences are false?

3 COMPLEX SENTENCES

Complex sentences of English generally translate into complex sentences of the logical notation. As usual, it is important to be clear about the grouping of clauses in the English sentence.

The following sentence is a simple conjunction:

Polk and Quincy were presidents $P \wedge Q$

The following sentence is also a conjunction, one of whose conjuncts is a negation:

Polk, but not Quincy, was a president. $P \wedge \sim Q$

This is a negation of a conjunction:

Not both Polk and Quincy were presidents. $\sim(P \wedge Q)$

This is a simple disjunction:

Either Polk or Quincy was president. $P \vee Q$

This is a complex sentence, with at least two different but equivalent symbolizations.

Neither Polk nor Quincy was president.

One symbolization is the negation of '*Either Polk or Quincy was president*; in this symbolization '*neither*' means '*not either*': $\sim(P \vee Q)$. An equivalent symbolization is a conjunction of negations; '*neither P nor Q*' is equivalent to "*not P and not Q*": $\sim P \wedge \sim Q$

The fundamental principles for our new connectives are:

> ***and, or, if and only if***
>
> When any of these expressions occurs between sentences, it gives rise to a conjunction, disjunction, or biconditional. The constituents of the conjunction, disjunction, or biconditional are symbolizations of sentences immediately to the left and to the right of '*and*', '*or*', or '*if and only if*'.
>
> When '*either*' occurs with '*or*', the symbolization of the expression enclosed between '*either*' and '*or*' is a disjunct. Likewise, When '*both*' occurs with '*and*', the symbolization of the expression enclosed between '*both*' and '*and*' is a conjunct.
>
> '*neither* □ *nor* ○' is equivalent to '*not* (*either* □ *or* ○)'.

As in chapter 1, these principles do not eliminate all ambiguity. The sentence '*Wilma will leave and Steve will stay or Tom will dance*' is ambiguous between these two symbolizations:

$W \,\&\, (S \vee T)$

$(W \& S) \vee T$

The use of '*either*' will sometimes disambiguate; the only symbolization of '*Wilma will leave and either Steve will stay or Tom will dance*' is:

$W \,\&\, (S \vee T)$

This is because '*either*' and '*or*' exactly enclose '*Steve will stay*', and so 'S' must be a disjunct. But it is not a disjunct in '$(W \& S) \vee T$'.

Commas play their usual role of grouping items on each side. The sentence '*Wilma will leave and Steve will stay, or Tom will dance*' has only the symbolization:

(W&S) ∨ T

Conjunction and disjunction signs inside of sentences: Sometimes '*and*' and '*or*' occur within sentences, as in:

Wilma sang and danced
Tom or Sam left

In such cases you need to fill in a missing part to get a sentence that we already know how to symbolize.

> Sometimes '*and*' or '*or*' occurs inside a simple sentence, where only the subject is conjoined or disjoined, and there is a single predicate, or only the predicate is conjoined or disjoined, and there is a single subject. If you fill in a copy of the shared part, you will get a synonymous sentence that we already know how to symbolize.

These are some examples:

Wilma sang and danced	*Wilma sang and [Wilma] danced*
Tom or Sam left	*Tom [left] or Sam left*

If there is a '*both*' or an '*either*, it ends up on the front:

Both Tom and Sam left	*Both Tom [left] and Sam left*
Either Tom or Sam left	*Either Tom [left] or Sam left*
Wilma both sang and danced	*Both Wilma sang and [Wilma] danced*
Wilma either sang or danced	*Either Wilma sang or [Wilma] danced.*

There may also be a 'not' after the compound subject, or before a compound predicate. If the negation is after a compound subject, it forms part of the predicate, and it is filled in with that predicate:

Wilma or Veronica didn't sing	*Wilma [didn't sing] or Veronica didn't sing.*

If the negation is before a compound predicate, it yields a negation sign that applies to the whole compound:

Wilma didn't sing or dance	¬(*Wilma sang or danced*)
Wilma didn't sing and dance	¬(*Wilma sang and danced*)

The parts inside the parentheses are then expanded as usual:

Wilma didn't sing or dance	¬(*Wilma sang or [Wilma] danced*)
Wilma didn't sing and dance	¬(*Wilma sang and [Wilma] danced)*

Compounds within simple sentences affect how sentences are grouped after symbolization:

> When connectives occur inside otherwise simple sentences, the symbolizations of the sentences form a unit.

For example, the sentence '*Ruth tap-dances or sings and she plays the clarinet*' must be grouped like this:

(T ∨ S) & P

This is because the disjunction with 'T' and 'S' must be a unit. In '*Ruth tap-dances or she sings and plays the clarinet*' the opposite happens; you must have:

T ∨ (S & P)

because the conjunction with 'S' and 'P' must form a unit.

> Synonyms of '*and*', '*or*', and '*if and only it*' are subject to the conditions described above.

Here are some illustrations:

If neither Wilma nor Sally attends, either Robert or Peter will be bored.

If neither Wilma [attends] nor Sally attends, either Robert [will be bored] or Peter will be bored.

If neither W *nor* S*, either* R *or* P

$\sim(W\vee S) \to (R\vee P)$

The '*neither*' and the '*either*' made units, and the comma was redundant.

A slightly more complex case:

If neither Wilma nor Sally attends, either Robert or Peter, but not Tom, will be bored.

If neither Wilma [attends] nor Sally attends, either Robert [will be bored] or Peter [will be bored], but Tom will not be bored.

If neither W *nor* S*, either* R *or* P*, but not* T

$\sim(W\vee S) \to (R\vee P)\ \&\ \sim T$

Here the '*either*' made a unit, so the '*but* ' could not take '*Peter would be bored*' as its left conjunct. That is, the symbolization could not be this:

$\sim(W\vee S) \to (R \vee (P\&\sim T))$

Likewise, the original phrase '*either Robert or Peter, but not Tom, will be bored*' consists of three simple sentences all sharing the '*will be bored*', so it could not be pulled apart, as in:

$(\sim(W\vee S) \to (R\vee P))\ \&\ \sim T$

Some additional examples:

Either Robert or Tom will attend, but not both
Either Robert [will attend] or Tom will attend, but not both [will attend]
Either R *or* T, *but not* R *and* T
$(R\vee T) \wedge\sim(R\wedge T)$

Robert will attend if Sally does, but she won't attend if neither Tom nor Wilma attend.
Robert will attend if Sally does [attend], but she won't attend if neither Tom [attends] nor Wilma attends.
R *if* S, *but not* S *if neither* T *nor* W
$(S\to R) \wedge(\sim(T\vee W) \to \sim S)$

Neither Sally nor Robert will run, but if either Tom or Quincy run, Veronica will win.
Neither S *nor* R, *but if either* T *or* Q, V
$\sim(S\vee R) \wedge(T\vee Q \to V)$.

Given that Sally and Robert won't both run, Tom will run exactly if Q does.
Given that not both S *and* R, T *exactly if* Q.
$\sim(S\wedge R) \to (T\leftrightarrow Q)$

A variety of English expressions that we have not mentioned affect how a sentence is to be symbolized. Examples:

Quincy will whistle if Reggie sings without Susan singing or Susan sings without Reggie, but he won't whistle if they both sing

Q *if* R *and not* S *or* S *and not* R, *but not* Q *if* S *and* R
$((R \wedge {\sim}S) \vee (S \wedge {\sim}R) \rightarrow Q) \wedge (S \wedge R \rightarrow {\sim}Q)$

Here '*Reggie sings without Susan singing*' means that Reggie sings and Susan doesn't sing.

If Sally runs, Rob will run, in which case Theodore will leave
$(S \rightarrow R) \wedge (R \rightarrow T)$

Here ' *in which case*' means "if Rob runs".

If a symbolization of a sentence is a correct one, then it and the English sentence being symbolized must agree in truth value no matter what truth values the simple sentences have. If they agree for every assignment of truth values, then the symbolization is correct. If not, it is incorrect. (To tell whether an English sentence is true or false given a specification of truth values for its simple parts you must rely on your understanding of English. To tell whether a symbolic sentence is true or false given the truth values of its sentential letters, you parse it and figure out its truth value as in section 1.)

EXERCISES

1. If 'P' is true and both 'Q' and 'R' are false, what are the truth values of the following? (In answering, give a parse tree for the sentence.)

a. ${\sim}(P \vee (Q \wedge R))$
b. ${\sim}P \vee (Q \wedge R)$
c. ${\sim}(P \vee R) \leftrightarrow {\sim}P \vee R$
d. ${\sim}Q \wedge (P \vee (Q \leftrightarrow R))$
e. $P \rightarrow ({\sim}Q \leftrightarrow ({\sim}R \rightarrow Q))$

For questions 2 and 3, use this translation scheme:

V	*Veronica will leave*
W	*William will leave*
Y	*Yolanda will leave*

2. For each of the following say which of the proposed translations is correct.

a. *Veronica won't leave if and only if William won't leave*
$${\sim}(V \leftrightarrow {\sim}W)$$
$${\sim}V \leftrightarrow {\sim}W$$
$$V \leftrightarrow {\sim}{\sim}W$$

b. *William and Veronica will both leave if Yolanda does, provided that Veronica doesn't*
$$Y \wedge {\sim}V \rightarrow W \wedge V$$
$$(Y \rightarrow W \wedge V) \rightarrow {\sim}V$$
$${\sim}V \rightarrow (Y \rightarrow W \wedge V)$$

c. *Unless Yolanda leaves, Veronica or William will leave*
$$Y \vee (W \vee V)$$
$$Y \rightarrow (W \vee V)$$
$$Y \leftrightarrow W \wedge V$$

d. *Either Yolanda leaves and Veronica doesn't, or Veronica leaves and William doesn't*
$$(Y \leftrightarrow {\sim}V) \vee (V \leftrightarrow {\sim}W)$$
$$(Y \wedge {\sim}V) \vee (V \wedge {\sim}W)$$
$$Y \wedge {\sim}V \leftrightarrow V \wedge {\sim}W$$

3. For each of the following produce a correct symbolization

a. *Only if Veronica doesn't leave will William leave, or Veronica and William and Yolanda will all leave*
b. *If neither William nor Veronica leaves, Yolanda won't either*
c. *If William will leave if Veronica leaves, then he will surely leave if Yolanda leaves*
d. *Neither William nor Veronica nor Yolanda will leave*

4. What are the truth values of 3a-d if Veronica leaves but neither William nor Yolanda leaves?

For question 5 use this translation scheme:

R	*Sally will run*
W	*Sally will win*
Q	*Sally will quit*

5. For each of the following produce a correct symbolization

a. *Sally will run and win unless she quits*
b. *Sally will win exactly in case she runs without quitting*
c. *Sally, who will run, will win if she doesn't quit*
d. *Sally will run and quit, but she will win anyway*

4 RULES

Each new connective comes with two new rules. As earlier, it should be obvious from the truth-table descriptions of each connective that instances of these rules are formally valid arguments.

Conjunction rules:

Rule s (simplification)

$\square \land \bigcirc$ or $\square \land \bigcirc$
$\therefore\ \square$ $\therefore\ \bigcirc$

Rule adj (adjunction)

$\square$
$\bigcirc$
$\therefore\ \square \land \bigcirc$

Disjunction rules:

Rule add (addition)

$\square$ or $\square$
$\therefore\ \square \lor \bigcirc$ $\therefore\ \bigcirc \lor \square$

Rule mtp (modus tollendo ponens)

$\square \lor \bigcirc$ or $\square \lor \bigcirc$
$\sim\bigcirc$ $\sim\square$
$\therefore\ \square$ $\therefore\ \bigcirc$

Biconditional rules:

Rule bc (biconditional-to-conditional)

$\square \leftrightarrow \bigcirc$ or $\square \leftrightarrow \bigcirc$
$\therefore\ \square \rightarrow \bigcirc$ $\therefore\ \bigcirc \rightarrow \square$

Rule cb (conditionals-to-biconditional)

$\square \rightarrow \bigcirc$
$\bigcirc \rightarrow \square$
$\therefore\ \square \leftrightarrow \bigcirc$

Simplification indicates that if you have a conjunction, you may infer either conjunct. For example, both of these valid arguments are instances of rule s:

Polk was a president and so was Whitney	$P \land W$	
$\therefore$ *Polk was a president*	$\therefore\ P$	by rule **s**
$\therefore$ *Whitney was a president*	$\therefore\ W$	by rule **s**

Adjunction indicates that if you have any two sentences, you may infer their conjunction, in either order. For example, these valid arguments are instances of rule adj:

Polk was a president	P	
Whitney was a president	W	
$\therefore$ *Polk was a president and so was Whitney*	$\therefore\ P \land W$	by rule **adj**
$\therefore$ *Whitney was a president and so was Polk*	$\therefore\ W \land P$	by rule **adj**

This derivation illustrates how the conjunction rules are used:

$P \land Q$
$\therefore\ Q \land P$

1. ~~Show~~ $Q \land P$
2. Q — pr1 s
3. P — pr1 s
4. $Q \land P$ — 2 3 adj dd

Addition indicates that from any sentence you may infer its disjunction with any other sentence.

Polk was a president	P	
$\therefore$ Polk was a president or Whitney was	$\therefore\ P \lor W$	by rule **add**
$\therefore$ Whitney was a president or Polk was	$\therefore\ W \lor P$	by rule **add**

Rule add lets you add any disjunct, no matter how irrelevant. So from '*Cynthia left*' you may infer '*Cynthia left* ∨ *Fido barked*'. This is legitimate because '∨' is used inclusively, and all that you need for a disjunction to be true is that either disjunct be true. So if '*Cynthia left*' is true, '*Cynthia left* ∨ *Fido barked*' must be true too.

Modus tollendo ponens indicates that from a disjunction and the negation of one of its disjuncts you may infer the other disjunct.

	Polk was a president or Whitney was		P ∨ W	
	Whitney wasn't a president		~W	
∴	Polk was a president	∴	P	by rule **mtp**

	Polk was a president or Whitney was		P ∨ W	
	Polk wasn't a president		~P	
∴	Whitney was a president	∴	W	by rule **mtp**

Note that the following is <u>not</u> an instance of modus tollendo ponens:

	Whitney was a president or Truman was		W ∨ T
	Truman was a president		T
∴	*Whitney wasn't a president*	∴	~W

For mtp you need the negation of a disjunct. In the case given, if 'T' and 'W' were both true, then the argument would have true premises and a false conclusion.

Here is a derivation illustrating the disjunction rules. It is a derivation for this argument:

P
R ∨ ~P
∴ R ∨ S

1. ~~Show~~ R ∨ S

2.	~~P	pr1 dn
3.	R	2 pr2 mtp
4.	R ∨ S	3 add dd

Biconditional-to-conditional indicates that from a biconditional you may infer either of the corresponding conditionals:

	Polk was a president if and only if Whitney was		P ↔ W	
∴	If Polk was a president, so was Whitney	∴	P → W	by rule **bc**
∴	If Whitney was a president, so was Polk	∴	W → P	by rule **bc**

Conditionals-to-biconditional indicates that from two conditionals where the antecedent of one is the consequent of the other, and vice versa, you may infer a bicondtional containing the parts of the conditionals:

	If Polk was a president, so was Whitney		P → W	
	If Whitney was a president, so was Polk		W → P	
∴	Polk was a president if and only if Whitney was	∴	P ↔ W	by rule **cb**

Here are two more derivations using our new rules:

S ∧ P
P∨Q → ~R
Q ∨ R
∴ Q

1. ~~Show~~ Q

2.	P	pr1 s
3.	P ∨ Q	2 **add**
4.	~R	3 pr2 mp
5.	Q	4 pr3 **mtp** dd

R ↔ ~P
~Q ↔ R
∴ P ↔ Q

1.	~~Show~~ P ↔ Q	
2.	~~Show~~ P → Q	
3.	P	ass cd
4.	~~P	3 dn
5.	R → ~P	pr1 **bc**
6.	~R	4 5 mt
7.	~Q → R	pr2 **bc**
8.	~~Q	6 7 mt
9.	Q	8 dn cd
10.	~~Show~~ Q → P	
11.	Q	ass cd
12.	~~Q	11 dn
13.	R → ~Q	pr2 **bc**
14.	~R	12 13 mt
15.	~P → R	pr1 **bc**
16.	~~P	14 15 mt
17.	P	16 dn cd
18.	P ↔ Q	2 10 **cb** dd

EXERCISES

1. For each of the following arguments, say which rule it is an instance of (or say "none").

a. P ∨ ~Q
Q
∴ P

b. ~P ∧ Q
∴ ~P

c. ~~(P→Q)
∴ P→Q

d. ~P∨Q
~Q
∴ ~P

e. ~P → ~Q
~Q → ~P
∴ ~Q ↔ ~P

f. P ∨ Q
~R
∴ P

g. ~~P ↔ R
∴ R → ~~P

h. Q
∴ ~P ∨ Q

i. P ∨ Q
∴ Q

2. Given the sentences below, say what can be inferred in one step by s, mtp, bc, cb using all of the premises.

a. ~W → ~X
~X → ~W
∴ ?

b. ~W ∨ ~X
~~X
∴ ?

c. W → X
~W
∴ ?

d. ~W ∧ ~X
∴ ?

e. W ↔ ~X
∴ ?

f. W ∨ X
∴ ?

5 SOME DERIVATIONS USING RULES S, ADJ, CB

Since there are new connectives it is useful to expand our strategy hints from Chapter 1:

Additional Strategy Hints

When trying to derive a conjunction, derive the conjuncts and then use adj.

When trying to derive a biconditional, derive the corresponding conditionals and use cb.

These strategy hints will be put to use below, as we extend our list of Theorems from Chapter 1.

Theorem 24 is the *commutative law* for conjunction; it says that turning the conjuncts of a sentence around produces a logically equivalent sentence:

T24 **$P \wedge Q \leftrightarrow Q \wedge P$** **"**commutative law for conjunction"

This is easy to derive if you follow the last two strategy hints. You will be deriving a biconditional, so you will try to derive both conditionals: $P \wedge Q \rightarrow Q \wedge P$ and $Q \wedge P \rightarrow P \wedge Q$ and then combine them using rule cb. While deriving each conditional you will derive a conjunction by deriving its conjuncts and then using rule adj. Rule s is used whenever you want to get one of the conjuncts of an existing conjunction alone.

1.	~~Show~~ $P \wedge Q \leftrightarrow Q \wedge P$	
2.	~~Show~~ $P \wedge Q \rightarrow Q \wedge P$	
3.	$P \wedge Q$	ass cd
4.	P	3 s
5.	Q	3 s
6.	$Q \wedge P$	4 5 adj cd
7.	~~Show~~ $Q \wedge P \rightarrow P \wedge Q$	
8.	$Q \wedge P$	ass cd
9.	Q	8 s
10.	P	8 s
11.	$P \wedge Q$	9 10 adj cd
12.	$P \wedge Q \leftrightarrow Q \wedge P$	2 7 cb dd

The next theorem is the *associative law* for conjunction; it says that regrouping successive conjuncts produces a logically equivalent sentence. The strategy here is the same as that above: to derive the biconditional you derive the corresponding conditionals, and use rule cb. In deriving the conditionals you derive conjunctions using rule adj. Again, rule s is used to simplify conjunctions that you already have.

T25 $\mathbf{P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R}$ "associative law for conjunction"

1.	~~Show~~ $P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$	
2.	~~Show~~ $P \wedge (Q \wedge R) \rightarrow (P \wedge Q) \wedge R$	
3.	$P \wedge (Q \wedge R)$	ass cd
4.	P	3 s
5.	$Q \wedge R$	3 s
6.	Q	5 s
7.	R	5 s
8.	$P \wedge Q$	4 6 adj
9.	$(P \wedge Q) \wedge R$	7 8 adj cd
10.	~~Show~~ $(P \wedge Q) \wedge R \rightarrow P \wedge (Q \wedge R)$	
11.	$(P \wedge Q) \wedge R$	ass cd
12.	R	3 s
13.	$P \wedge Q$	3 s
14.	P	5 s
15.	Q	5 s
16.	$Q \wedge R$	4 6 adj
17.	$P \wedge (Q \wedge R)$	7 8 adj cd
18.	$P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$	2 10 cb dd

The next theorem is T26:

T26 $\mathbf{(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)}$ "hypothetical syllogism"

Notice that **T26** and **T4** from the previous chapter are both called "hypothetical syllogism".

T4 $\mathbf{(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))}$ "hypothetical syllogism"

These two theorems are closely related. They are related to one another as the following two patterns:

$\square \wedge \bigcirc \rightarrow \triangle$
$\square \rightarrow (\bigcirc \rightarrow \triangle)$

where each theorem has 'P→Q' in place of □, 'Q→R' in place of ○, and 'P→R' in place of △. Our next theorem says that these two patterns are equivalent.

T27 $\mathbf{(P \wedge Q \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))}$ "exportation"

The derivation of this theorem is also relatively straightforward: derive two conditionals and put them together by rule cb. Each conditional itself has conditionals as parts, so the derivation calls for two conditional subderivations (one of which itself contains another conditional subderivation).

1.	~~Show~~ $(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	
2.	~~Show~~ $(P\wedge Q \to R) \to (P\to (Q\to R))$	
3.	$P\wedge Q \to R$	ass cd
4.	~~Show~~ $P\to (Q\to R)$	
5.	P	ass cd
6.	~~Show~~ $Q \to R$	
7.	Q	ass cd
8.	$P \wedge Q$	5 7 adj
9.	R	3 8 mp cd
10.		6 cd
11.		4 cd
12.	~~Show~~ $(P\to (Q\to R)) \to (P\wedge Q \to R)$	
13.	$P \to (Q\to R)$	ass cd
14.	~~Show~~ $P\wedge Q \to R$	
15.	$P \wedge Q$	ass cd
16.	P	15 s
17.	$Q \to R$	13 16 mp
18.	Q	15 s
19.	R	17 18 mp cd
20.		14 cd
21.	$(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	2 12 cb dd

This derivation is complex. It may be useful to see how we might think up how to construct it. First, our main strategy is to derive a biconditional by deriving two conditionals. So our plan predicts that the derivation will have this overall structure:

1.	~~Show~~ $(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	
2.	$(P\wedge Q \to R) \to (P\to (Q\to R))$	
12.	$(P\to (Q\to R)) \to (P\wedge Q \to R)$	
21.	$(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	2 12 cb dd

Line 2 will require a conditional derivation, and so will line 12. So the completed derivation will take this form:

1.	~~Show~~ $(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	
2.	~~Show~~ $(P\wedge Q \to R) \to (P\to (Q\to R))$	
3.	$P\wedge Q \to R$	ass cd
	$P \to (Q\to R)$	xxx cd
12.	~~Show~~ $(P\to (Q\to R)) \to (P\wedge Q \to R)$	
13.	$P \to (Q\to R)$	ass cd
	$P\wedge Q \to R$	xxx cd
21.	$(P\wedge Q \to R) \leftrightarrow (P\to (Q\to R))$	2 12 cb dd

Lines 3-11 and 14-20 are taken up with completing the subderivations. Each of these itself uses a conditional subderivation, giving the following structure:

1.	~~Show~~ $(P \wedge Q \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$	
2.	~~Show~~ $(P \wedge Q \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R))$	
3.	$P \wedge Q \rightarrow R$	ass cd
4.	~~Show~~ $P \rightarrow (Q \rightarrow R)$	
5.	P	ass cd
10.		xxx cd
11.		4 cd
12.	~~Show~~ $(P \rightarrow (Q \rightarrow R) \rightarrow (P \wedge Q \rightarrow R)$	
13.	$P \rightarrow (Q \rightarrow R)$	ass cd
14.	~~Show~~ $P \wedge Q \rightarrow R$	
15.	$P \wedge Q$	ass cd
19.		xxx cd
20.		14 cd
21.	$(P \wedge Q \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$	2 12 cb dd

The rest of the work is filling in the remaining subderivations. It is often useful to develop a derivation as we did here by first sketching its overall structure, and then flesh it out with details afterwards.

EXERCISES

1. Produce derivations for theorems T28-T30, T33, T36-37, which are included among the theorems stated here:

T28	$\mathbf{(P \wedge Q \rightarrow R) \leftrightarrow (P \wedge \sim R \rightarrow \sim Q)}$	
T29	$\mathbf{(P \rightarrow Q \wedge R) \leftrightarrow (P \rightarrow Q) \wedge (P \rightarrow R)}$	"distribution of $\rightarrow$ over $\wedge$"
T30	$\mathbf{(P \rightarrow Q) \rightarrow (R \wedge P \rightarrow R \wedge Q)}$	
T31	$\mathbf{(P \rightarrow Q) \rightarrow (P \wedge R \rightarrow Q \wedge R)}$	
T32	$\mathbf{(P \rightarrow R) \wedge (Q \rightarrow S) \rightarrow (P \wedge Q \rightarrow R \wedge S)}$	"Leibniz's *praeclarum theorema*"
T33	$\mathbf{(P \rightarrow Q) \wedge (\sim P \rightarrow Q) \rightarrow Q}$	"separation of cases; constructive dilemma"
T34	$\mathbf{(P \rightarrow Q) \wedge (P \rightarrow \sim Q) \rightarrow \sim P}$	"reductio ad absurdum"
T35	$\mathbf{(\sim P \rightarrow R) \wedge (Q \rightarrow R) \leftrightarrow ((P \rightarrow Q) \rightarrow R)}$	
T36	$\mathbf{\sim (P \wedge \sim P)}$	"non-contradiction"
T37	$\mathbf{(P \rightarrow Q) \leftrightarrow \sim (P \wedge \sim Q)}$	

6 ABBREVIATING DERIVATIONS

It is useful in writing derivations to be able to combine two or more steps into one. For example, here is a derivation in which double negation is used twice:

P
~Q → ~P
Q → R
∴ R

1. ~~Show~~ R
2. ~~P   pr1 dn
3. ~~Q   2 pr2 mt
4. Q   3 dn
5. R   4 pr3 mp dd

One can shorten this derivation by two steps by combining the double negations with other rules, like this:

1. ~~Show~~ R
2. ~~Q   pr1 dn pr2 mt
3. R   2 dn pr3 mp dd

The meanings of the notations at the end of the lines are:

2. " pr1 dn pr2 mt "   take pr1 and double negate it; then combine the result with pr2 by mt to get ~~Q
3. "2 dn pr3 mp "   double (un)negate the sentence on line 2; then combine the result with pr3 using modus ponens

Here is a more highly abbreviated derivation.

P ∧ Q
R → ~Q
S∨~R → T
∴ T ∧ P

1. ~~Show~~ T ∧ P

2.	~R	pr1 s dn pr2 mt	simplify pr1 to get Q, then double negate Q to get ~~Q; use **mt** on this and pr2 to get ~R
3.	T	2 add pr3 mp	apply **add** to the sentence on line 2 to get S ∨ ~R; then apply mp to that and pr3 to get T
4.	T ∧ P	pr1 s 3 adj dd	simplify pr1 to get P and then adjoin this with the sentence on line 3 to get T∧P.

Abbreviations of this sort may always be interpreted by the following "decoding procedure", starting at the left and moving right:

A line number or premise number gives you a sentence -- the sentence on that line.

Rule r also gives you a sentence -- the sentence on the line cited.

A sentence followed by 'dn', 's', 'add' or 'bc' gives you the result of applying that rule to that sentence. (The old sentence is no longer available for further use.)

Two sentences followed by 'mp', 'mt', 'adj', 'cb' give you the result of applying that rule to them.

If you can apply this decoding and end up with the sentence on the line which has the abbreviations at its end, the line is correct. If you can't, the line is not correct. (There is sometimes more than one way to apply a rule to a sentence, so there may be many ways to use the decoding process. If at least one way of using it ends you up with the sentence on the line, the abbreviation is correct; otherwise it is incorrect.)

Applied to the abbreviations on lines 2, 3 and 4 above the decoding looks like this. We work from the left. First, the leftmost 'pr1' is replaced by the first premise:

2. pr1 s dn pr2 mt

2. $P \wedge Q$ s dn pr2 mt

Then rule s acts on this to produce 'Q':

2. $P \wedge Q$ s dn pr2 mt

2. Q dn pr2 mt

Then double negation turns 'Q' into '~~Q':

2. Q dn pr2 mt

2. ~~Q pr2 mt

Then pr2 is replaced by the second premise:

2. ~~Q pr2 mt

2. ~~Q $R \rightarrow$ ~Q mt

Finally, rule mt acts on '~~Q' and '$R \rightarrow$ ~Q' to give you '~R', which is the sentence that actually appears on line 2:

2. ~~Q pr2 mt

2. ~R

Line 3 is decoded by the same process:

3. 2 add pr3 mp

3. ~R add pr3 mp

3. $S \vee$ ~R pr3 mp

3. $S \vee$ ~R $S \vee$~$R \rightarrow T$ mp

3. T

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Likewise for line 4:

4.	pr1	s	3	adj
4.	P∧Q	s	3	adj
4.		P	3	adj
4.		P	T	adj
4.			T∧P	

A long string of abbreviations can be difficult to decode, so we will confine ourselves to simple cases.

EXERCISES

1. Use the method of abbreviating derivations to produce shortened derivations for T38, T40-43.

T38	**P∧Q ↔ ~(P → ~Q)**	
T39	**~(P∧Q) ↔ (P → ~Q)**	
T40	**~(P → Q) ↔ P∧~Q**	"negation of conditional"
T41	**P ↔ P∧P**	"idempotence for ∧"
T42	**P∧~Q → ~(P→Q)**	"negation of conditional"
T43	**~P → ~(P∧Q)**	
T44	**~Q → ~(P∧Q)**	

7 USING THEOREMS AS RULES

In Chapter 1 we learned a way to introduce instances of previously derived theorems into a derivation. Theorems are even more useful when they are used to justify rules. The fundamental principle is that if a theorem has been derived that has the form of a conditional, it can be cited as a rule which allows you to infer one sentence from another whenever the conditional made from those sentences is an instance of the theorem.

> A theorem of conditional form:
>
> $\therefore\ \Box \to \bigcirc$
>
> justifies a rule of this form:
>
> $\Box$
> $\therefore\ \bigcirc$

For example, T13 ("transposition") is $(P \to Q) \to (\sim Q \to \sim P)$. This validates the rule:

$\Box \to \bigcirc$
$\therefore\ \sim\bigcirc \to \sim\Box$

We name such a rule by writing 'R' in front of the name of the theorem being used. An example of a use of a theorem as a rule is:

8.	$S \to T$	
9.	$\sim T \to \sim S$	8 RT13

Here are two arguments, and derivations, that use some theorems from Chapter 1 as rules.

$\sim(Q \land \sim R) \to P$
$P \to Q$
$R \to \sim P$
$\therefore\ \sim(Q \to R)$

1.	~~Show~~ $\sim(Q \to R)$		
2.	$Q \to R$	ass id	
3.	$(Q \to R) \to (P \to R)$	pr2 **RT4**	T4 is $(P \to Q) \to ((Q \to R) \to (P \to R))$
4.	$P \to R$	2 3 mp	
5.	$(R \to \sim P) \to (P \to \sim P)$	4 **RT4**	
6.	$P \to \sim P$	pr3 5 mp	
7.	$\sim P$	6 **RT20**	T20 is $(P \to \sim P) \to \sim P$
8.	$\sim\sim(Q \land \sim R)$	pr1 7 mt	
9.	$Q \land \sim R$	8 dn	
10.	Q	9 s	
11.	R	2 10 mp	
12.	$\sim R$	9 s 11 id	

$S \to T$
$T \to (Q \to P)$
$S \to Q$
$\therefore\ S \to P$

1.	~~Show~~ $S \to P$		
2.	$S \to (Q \to P)$	pr1 pr2 RT4	T4 is $(P \to Q) \to ((Q \to R) \to (P \to R))$
3.	$(S \to Q) \to (S \to P)$	2 RT6	T6 is $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
4.	$S \to P$	pr3 3 mp dd	

Theorems can be used to make rules in two more ways. One way applies when a theorem is a biconditional. Since a biconditional is logically equivalent to two conditionals, it makes sense to use the theorem as if it were two conditionals.

> A theorem of biconditional form:
>
> ∴ □ ↔ ○
>
> validates both of these rules:
>
> | | |
> |---|---|
> | □ | ○ |
> | ∴ ○ | ∴ □ |

An example is T27: '(P∧Q→R) ↔ (P→(Q→R))' which validates both of these:

RT27	□∧○ → △	RT27	□ → (○→△)
	∴ □ → (○→△)		∴ □∧○ → △

A final additional way to use theorems as rules is possible when a theorem is a conditional whose antecedent is a conjunction. This gives a rule which has multiple premises.

> A theorem of this form:
>
> ∴ □∧○ → △
>
> justifies a rule of this form:
>
> □
> ○
> ∴ △

An example is T26: (P→Q) ∧(Q→R) → (P→R)

which validates:

RT26
□ → ○
○ → △
∴ □ → △

These options combine, so that if one side of a biconditional is a conjunction, it validates a rule with multiple premises. For example, T38 (below) is 'P∧Q ↔ ~(P→~Q)', and one of the rules that it validates is:

RT38
□
○
∴ ~(□→~○)

EXERCISES

1. For each of the following derivations, determine which lines are correct and which incorrect. (In assessing a line, assume that previous lines are correct.)

$\therefore\ ((U\to V) \to S) \to (\sim S \to U)$

1.	~~Show~~ $((U\to V) \to S) \to (\sim S \to U)$		
2.	$(U\to V) \to S$	ass cd	
3.	~~Show~~ $\sim S \to U$		
4.	$\sim S$	ass cd	
5.	$\sim S \to \sim(U\to V)$	2 RT13	T13 is $(P\to Q) \to (\sim Q \to \sim P)$
6.	$\sim(U \to V)$	4 5 mp	
7.	U	6 RT21 cd	T21 is $\sim(P\to Q) \to P$
8.		3 cd	

$\therefore\ \sim V \wedge (W \to V\wedge U) \to (\sim W \to \sim S)$

1.	~~Show~~ $\sim V \wedge (W \to V\wedge U) \to (\sim W \to \sim S)$		
2.	$\sim V \wedge (W \to V\wedge U)$	ass cd	
3.	$\sim V$	2 s	
4.	$\sim(V\wedge U)$	3 RT43	T43 is $\sim P \to \sim(P\wedge Q)$
5.	$W \to V\wedge U$	2 s	
6.	$\sim W$	4 5 mt	
7.	$\sim(W\wedge S)$	6 RT43	
8.	$\sim W \to \sim S$	7 RT39 cd	T39 is $\sim(P\wedge Q) \leftrightarrow (P\to\sim Q)$

Some more theorems:

T45 $P\vee Q \leftrightarrow (\sim P\to Q)$

T46 $(P\to Q) \leftrightarrow \sim P \vee Q$ "definition of $\to$ in terms of $\vee$"

T47 $P \leftrightarrow P\vee P$ "idempotence for $\vee$"

T48 $(P\vee Q) \wedge(P\to R) \wedge(Q\to S) \to R\vee S$

T49 $(P\vee Q) \wedge(P\to R) \wedge(Q\to R) \to R$ "separation of cases"

T50 $(P\to R) \wedge(Q\to R) \leftrightarrow (P\vee Q\to R)$

2. Construct a derivation for T45, and then use RT45 to derive T46

3. Construct derivations for T47 and T48, and construct a derivation for T49 using RT47 and RT48

4. Use RT49 in constructing a derivation for T50.

5. Derive T53.

Some additional theorems are given here for reference.

T51 $(P\vee Q) \wedge(P\to R) \wedge(\sim P\wedge Q\to R) \to R$

T52 $(P\to R) \wedge(\sim P\wedge Q\to R) \leftrightarrow (P\vee Q\to R)$

T53	**P∨Q ↔ Q∨P**	"commutative law for ∨"
T54	**P ∨ (Q∨R) ↔ (P∨QR) ∨ R**	"associative law for ∨"
T55	**(P → Q∨R) ↔ (P→Q) ∨ (P→R)**	"distribution of → over ∨"
T56	**(P→Q) → (R∨P → R∨Q)**	
T57	**(P→Q) → (P∨R → Q∨R)**	
T58	**(P→Q) ∨ (Q→R)**	
T59	**P ∨ ~P**	"excluded middle"
T60	**(P→R) ∨ (Q→R) ↔ (P∧Q → R)**	
T61	**P∧ (Q∨R) ↔ (P∧Q) ∨ (P∧R)**	"distribution"
T62	**P∨(Q∧R) ↔ (P∨Q) ∧(P∨R)**	"distribution"

8 DERIVED RULES

We now have over fifty theorems that can be used as rules, with more to come. There are too many of these to remember easily. It is customary to isolate a small number of rules based on the theorems and give them special names, and use these rules frequently in derivations. In this section we look at five of these.

Rule nc (negation of conditional): 'Negation of conditional' is an easy-to-remember name for rule RT40 (which is "~(P→Q) ↔ P∧~Q"). It applies in either of these forms:

Rule nc			
	~(□→○)		□ ∧ ~○
∴	□ ∧ ~○	∴	~(□→○)

This rule is often useful when you are trying to derive a conditional when a conditional derivation isn't working for you. Instead of assuming the antecedent of the conditional in order to use cd, assume the negation of the conditional for an indirect derivation. Then turn this negated conditional into a conjunction of the antecedent with the negation of its consequent. This gives you a lot to work with in continuing the derivation. As an example, suppose you are trying to validate this argument:

R → Q
R ∨ S
S → R
∴ P → Q

You begin the derivation:

1. Show P → Q

You may now assume 'P' for purposes of doing a conditional derivation. But 'P' does not occur among the premises, and you may not see how to proceed. Instead of trying a conditional derivation, begin an indirect derivation:

1. Show P → Q
2.   ~(P→Q)   ass id

Then apply the derived rule nc:

1.	Show $P \rightarrow Q$	
2.	$\sim(P\rightarrow Q)$	ass id
3.	$P \wedge \sim Q$	2 nc

It is now fairly easy to simplify off '~Q', and use it to derive a contradiction:

1.	Show $P \rightarrow Q$			
2.	$\sim(P\rightarrow Q)$	ass id		
3.	$P \wedge \sim Q$	2 nc		
4.	$\sim Q$	3 s		
5.	$\sim R$	4 pr1 mt	←	
6.	S	5 pr2 mtp		contradictories
7.	R	6 pr3 mp	←	

So you can finish the indirect derivation:

1.	~~Show~~ $P \rightarrow Q$	
2.	$\sim(P\rightarrow Q)$	ass id
3.	$P \wedge \sim Q$	2 nc
4.	$\sim Q$	3 s
5.	$\sim R$	4 pr1 mt
6.	S	5 pr2 mtp
7.	R	6 pr3 mp 5 id

Rule cdj (conditional as disjunction): This rule is constituted by T45 and T46, which together assert the equivalence of a conditional with a disjunction whose left disjunct is the negation (or unnegation) of the antecedent of the conditional and whose right disjunct is the consequent. Rule cdj has four cases:

Rule cdj

$\square \rightarrow \bigcirc$	$\sim\square \vee \bigcirc$	$\sim\square \rightarrow \bigcirc$	$\square \vee \bigcirc$
$\therefore\ \sim\square \vee \bigcirc$	$\therefore\ \square \rightarrow \bigcirc$	$\therefore\ \square \vee \bigcirc$	$\therefore\ \sim\square \rightarrow \bigcirc$

This rule can be useful when attempting to derive a disjunction. Instead of deriving the disjunction directly, derive the conditional whose antecedent is the (un)negation of the left disjunct and whose consequent is the other disjunct. This can usually be done using a conditional derivation. Then turn the result of the conditional derivation into the disjunction you are after using derived rule cdj.

Here is a derivation of T54 using cdj together with T53, which was derived in the exercises for the last section. The overall structure of the derivation is to derive the biconditional by using two conditional derivations to get the corresponding conditionals, and then use rule cb.

T54 $\quad$ **$P \lor (Q\lor R) \leftrightarrow (P\lor Q) \lor R$**

1. ~~Show~~ $P \lor (Q\lor R) \leftrightarrow (P\lor Q) \lor R$
2. ~~Show~~ $P \lor (Q\lor R) \to (P\lor Q) \lor R$
3. $P \lor (Q\lor R)$ — ass cd
4. ~~Show~~ $\sim R \to (P\lor Q)$
5. $\sim R$ — ass cd
6. ~~Show~~ $\sim P \to Q$
7. $\sim P$ — ass cd
8. $Q\lor R$ — 3 7 mtp
9. Q — 5 8 mtp cd
10. $P \lor Q$ — 6 **cdj** cd $\quad \leftarrow$ cdj
11. $R \lor (P\lor Q)$ — 4 **cdj** $\quad \leftarrow$ cdj
12. $(P\lor Q) \lor R$ — 11 RT53 cd
13. ~~Show~~ $(P\lor Q) \lor R \to P \lor (Q\lor R)$

    LIKE LINES 3-12 <with two uses of **cdj**> $\quad \leftarrow$ cdj twice

24. $P \lor (Q\lor R) \leftrightarrow (P\lor Q) \lor R$ — 2 13 cb

Rule sc (separation of cases): This is a combination of RT33 and RT49. It validates these inferences:

Rule sc

$\square \lor \bigcirc$	$\square \to \triangle$
$\square \to \triangle$	$\sim\square \to \triangle$
$\bigcirc \to \triangle$	$\therefore\ \triangle$
$\therefore\ \triangle$	

The first form of rule sc (on the left) says that if you are given that at least one of two cases hold (the first premise), and if each of them imply something (the second and third premises), then you can conclude that thing.

The second form of rule sc (on the right) applies when one of the two cases is the negation of the other. Then their disjunction ($P \lor \sim P$) is logically true, and needn't be stated as an additional premise. (See below for illustration.)

Rule sc is especially useful when other attempts to produce a derivation have failed. For example, if you have a disjunction on an available line, then see if you can do two conditional derivations, each starting with one of the disjuncts, and each reasoning to the desired conclusion. If you can do this, the first form of sc applies. As an example, suppose you are given this argument:

$V \lor W$
$W \to \sim X$
$\sim U \to X$
$\therefore\ U \lor V$

It may not be apparent how to proceed. So consider separation of cases. You have available a disjunction, '$V \lor W$', which is the first premise. If you can derive both $V \to U\lor V$ and $W \to U\lor V$, the rule sc will give you the desired conclusion:

1. ~~Show~~ $U \vee V$
2. ~~Show~~ $V \rightarrow U \vee V$
3. V ass cd
4. $U \vee V$ 3 add cd
5. ~~Show~~ $W \rightarrow U \vee V$
6. W ass cd
7. $\sim X$ 6 pr2 mp
8. U 7 pr3 mt dn
9. $U \vee V$ 8 add cd
10. $U \vee V$ pr1 2 5 sc dd

When you don't have a disjunction to work with, you may be able to use the second form of sc. Suppose you have this argument:

$R \wedge S \rightarrow Q$
$R \rightarrow S$
$\therefore\ R \rightarrow Q$

In applying the second form of sc, you need to choose something which will serve as the antecedent for a conditional whose consequent is the desired conclusion, and whose negation will also serve as the antecedent for a conditional whose consequent is the desired conclusion. What should you choose? Often there is more than one choice that will work. In the case we are given, 'R' will work for this purpose. That is, you will indeed be able to derive both of these:

$R \rightarrow (R \rightarrow Q)$
$\sim R \rightarrow (R \rightarrow Q)$

The second form of rule sc will then give you the desired conclusion:

1. ~~Show~~ $R \rightarrow Q$
2. ~~Show~~ $R \rightarrow (R \rightarrow Q)$ ← derive '$R \rightarrow (R \rightarrow Q)$'
3. R ass cd
4. S 3 pr2 mp
5. Q 3 4 adj pr1 mp cd
6. ~~Show~~ $\sim R \rightarrow (R \rightarrow Q)$ ← derive '$\sim R \rightarrow (R \rightarrow Q)$'
7. $\sim R$ ass cd
8. ~~Show~~ $R \rightarrow Q$
9. R ass cd
10. $\sim R$ 7 r id
11. 8 cd
12. $R \rightarrow Q$ 2 6 sc dd ← apply the second form of sc

Rule dm (DeMorgan's): This is a very useful rule. It lets you replace negations of conjunctions with modified disjunctions, and vice versa. It consists of any application of the rules based on theorems T63-T66:

T63 $\quad$ **$P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$**

T64 $\quad$ **$P \vee Q \leftrightarrow \sim(\sim P \wedge \sim Q)$**

T65 $\quad$ **$\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$**

T66 $\quad$ **$\sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$**

So it allows any of the following inferences:

Rule dm

$\square \wedge \bigcirc$	$\square \vee \bigcirc$	$\sim(\square \wedge \bigcirc)$	$\sim(\square \vee \bigcirc)$
$\therefore\ \sim(\sim\square \vee \sim\bigcirc)$	$\therefore\ \sim(\sim\square \wedge \sim\bigcirc)$	$\therefore \sim\square \vee \sim\bigcirc$	$\therefore \sim\square \wedge \sim\bigcirc$
$\sim(\sim\square \vee \sim\bigcirc)$	$\sim(\sim\square \wedge \sim\bigcirc)$	$\sim\square \vee \sim\bigcirc$	$\sim\square \wedge \sim\bigcirc$
$\therefore\ \square \wedge \bigcirc$	$\therefore\ \square \vee \bigcirc$	$\therefore\ \sim(\square \wedge \bigcirc)$	$\therefore\ \sim(\square \vee \bigcirc)$

It may be easiest to remember these forms by remembering T63 and T64 in this form:

A negation of a $\left\{\begin{matrix}\text{conjunction}\\ \text{disjunction}\end{matrix}\right\}$ is equivalent to the $\left\{\begin{matrix}\text{disjunction}\\ \text{conjunction}\end{matrix}\right\}$ of the negations of its parts.

DeMorgan's rule can be handy when you are trying to derive a disjunction. To use it, you assume the negation of the disjunction for an indirect derivation. Rule dm lets you turn that negation into a conjunction, and then you have both conjuncts to use in deriving a contradiction. Example:

$P \rightarrow U$
$P \vee Q$
$Q \rightarrow V$
$\therefore\ U \vee V$

1. ~~Show~~ $U \vee V$

2.	$\sim(U \vee V)$	ass id
3.	$\sim U \wedge \sim V$	2 dm
4.	$\sim P$	3 s pr1 mt
5.	Q	4 pr2 mtp
6.	V	5 pr3 mp
7.	$\sim V$	3 s 6 id

Rule nb (negation of biconditional): This rule is an application of T90:

$\sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$

The rule sanctions these inferences:

Rule nb

$\sim(\square \leftrightarrow \bigcirc)$	$\square \leftrightarrow \sim\bigcirc$
$\therefore\ \square \leftrightarrow \sim\bigcirc$	$\therefore\ \sim(\square \leftrightarrow \bigcirc)$

The first form is handy if you have the negation of a biconditional. The rule lets you infer a biconditional, which simplifies into two conditionals, which can be very useful. Here is an example:

$\sim(P \leftrightarrow Q)$
$\sim Q$
$\therefore P$

1. ~~Show~~ P

2.	$\sim(P \leftrightarrow Q)$	pr
3.	$P \leftrightarrow \sim Q$	2 nb
4.	$\sim Q \rightarrow P$	3 bc
5.	P	4 pr2 mp dd

The second form is handy if you want to derive the negation of a biconditional. Just derive the related biconditional, say by using conditional derivations to derive the associated conditionals. Example:

$P \rightarrow (R \leftrightarrow Q)$
$R \rightarrow \sim Q$
$S \rightarrow Q$
$\sim R \rightarrow S$
$\therefore \sim P$

1. ~~Show~~ ~P

2.	~~Show~~ $\sim Q \rightarrow R$	
3.	$\sim Q$	ass cd
4.	$\sim S$	3 pr3 mt
5.	$\sim\sim R$	4 pr4 mt
6.	R	5 dn cd
7.	$R \leftrightarrow \sim Q$	pr2 2 cb
8.	$\sim(R \leftrightarrow Q)$	7 nb
9.	$\sim P$	8 pr1 mt dd

EXERCISES

1. For each of the following derivations, determine which lines are correct and which incorrect. (In assessing a line, assume that previous lines are correct.)

a. $(U \rightarrow S) \rightarrow Q$
$P \vee R \rightarrow S$
$\sim(T \rightarrow Q)$
$\therefore \sim P$

1. ~~Show~~ ~P

2.	$T \wedge \sim Q$	pr3 nc
3.	$\sim(U \rightarrow S)$	2 s pr1 mt
4.	$U \wedge \sim S$	3 nc
5.	$\sim(P \vee R)$	4 s pr2 mt
6.	$\sim P \wedge \sim R$	5 dm
7.	$\sim P$	6 s dd

b. ~X ∨ W
~(V ↔ W)
~(W ↔X) ∨ V
∴ ~W

1. ~~Show~~ ~W
2. ~~Show~~ W → X
3. W ass cd
4. X 3 pr1 mtp
5. ~~Show~~ X → W
6. X ass cd
7. X → W pr1 cdj dd
8. W ↔ X 2 5 bc
9. V 8 dn pr3 mtp
10. V ↔ ~W pr2 nb
11. ~W 9 10 mp dd

c. (X →U) → (Y→Z)
~(Y ∨ ~Z)
∴ ~U

1. ~~Show~~ ~U
2. ~Y ∧ Z pr2 dm
3. ~(Y → Z) 2 nc
4. ~(X → U) 3 4 mt
5. X ∧ ~U 4 nc
6. ~U 5 s dd

2. Construct correct derivations for each of the following arguments using derived rules when convenient.

a. U∧V → X <use dm>
~V → Y
X∨Y → Z
∴ ~Z → ~U

b. (X→Y) → Z <use nc>
~Z
V → Y
∴ ~V

c. P ∨ Q
Q → S
U ∨ ~S
P∨S → R
R → U
∴ U

9 OFFICIAL CONDITIONS FOR DERIVATIONS

Let us summarize here what we can now use in constructing an unabbreviated derivation.

UNABBREVIATED DERIVATIONS

A **derivation** from a set of sentences P consists of a sequence of lines that is built up in order, step by step, where each step is in accordance with these provisions:

- **Show line:** A show line consists of the word "Show" followed by a symbolic sentence. A show line may be introduced at any step. Show lines are not given a justification.
- **Premise**: At any step, any symbolic sentence from the set P may be introduced, justified with the notation "pr".
- **Theorem**: At any step, an instance of a previously proved theorem may be entered with the name of the theorem given as justification. (e.g. "T32")
- **Rule:** At any step, a line may be introduced if it follows by a rule from sentences on previous available lines; it is justified by citing the numbers of those previous lines and the name of the rule. This includes the following basic rules:

r	mp	mt	dn	s
adj	add	mtp	bc	cb

  It also includes rules based on previously derived theorems, where the name of a rule based on a theorem is "R" followed by the name of the theorem; e.g. "RT32". If the appropriate enabling theorems have been derived, these rules are also available for use:

nc	cdj	sc	dm	nb
- **Direct derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the sentence on the closest previous uncancelled show line, one may, as the next step, write "dd" following the justification for that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for conditional derivation**: When a show line with a conditional sentence is introduced, as the next step one may introduce an immediately following line with the antecedent of the conditional on it; the justification is "ass cd".
- **Conditional derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the consequent of the conditional sentence on the closest previous uncancelled show line, one may, as the next step, write "cd" at the end of that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for indirect derivation**: When a show line is introduced, as the next step one may introduce an immediately following line with the [un]negation of the sentence on the show line; the justification is "ass id".
- **Indirect derivation:** When a sentence is introduced on a line which is not a show line, if there is a previous available line containing the [un]negation of that sentence, and if there is no uncancelled show line between the two sentences, as the next step you may write the line number of the first sentence followed by "id" at the end of the line with the second sentence. Then you cancel the closest previous "show", and box all sentences below that show line, including the current line.

Except for steps that involve boxing and canceling, every step introduces a line. When writing out a derivation, every line that is introduced is written directly below previously introduced lines.

Optional variant: When boxing and canceling with direct or conditional derivation, the "dd" or "cd" justification may be written on a later line which contains no sentence at all, and which is followed by the number of the line that completes the derivation. With indirect derivation, the "id" justification may be written on a later line which contains no sentence at all, and which is followed by the numbers of the two lines containing contradictory sentences. In all cases, the lines cited must be available from the later line.

Some additional strategic hints

Now that we have connectives in addition to the negation and conditional signs, we can give some general hints for doing derivations containing them. These have all been illustrated above, and they will simply be stated here for convenience. First are strategies that are often useful for deriving certain forms of sentences.

If you want to derive a **Conjunction** □ ∧ ○
Derive each conjunct (perhaps by id) and adjoin them

If you want to derive a **Disjunction** □ ∨ ○
Derive either disjunct and use add.
Assume '~(□ ∨ ○)' for id, and use dm.
Derive '~□ → ○', perhaps by cd, and use cdj

If you want to derive a **Biconditional** □ ↔ ○
Derive each conditional and use cb.

If you want to derive a **Negation of a conjunction ~(□ ∧ ○)**
Use id.

If you want to derive a **Negation of a disjunction ~(□ ∨ ○)**
Derive '~□ ∧ ~○' and use dm.
Perhaps assume '□ ∨ ○' for id, and try to derive both '□ → P∧~P' and '○→P∧~P'. Then use sc (applied to the assumed '□ ∨ ○' and the conditionals) to derive 'P∧~P'.

If you want to derive a **Negation of a biconditional ~(□ ↔ ○)**
Derive '□ ↔ ~○' and use nb.

Then there are situations in which you have available a certain form of sentence, and want to know how to make use of it.

If you have available a **Conjunction** □ ∧ ○
Simplify and use the conjuncts singly.

If you have available a **Disjunction** □ ∨ ○
Try to derive the negation of one of the disjuncts, and use mtp
Derive the conditionals '□ → △' and '□ → △', where '△' is something you want to derive. Then use sc with the disjunction and two conditionals.

If you have available a **Biconditional** □ ↔ ○
Infer both conditionals and use them with mp, mt, and so on.

If you have available a **Negation of a conjunction ~(□ ∧ ○)**
Use dm to turn this into '~□ ∨ ~○', and then try to derive either '□' or '○' to use mtp.

If you have available a **Negation of a disjunction ~(□ ∨ ○)**
Use dm to turn this into '~□ ∧ ~○'; then simplify and use the conjuncts singly.

If you have available a **Negation of a biconditional ~(□ ↔ ○)**
Use nb to turn this into '□ ↔ ~○', and use bc to get the corresponding conditionals.

EXERCISES

1. Construct correct derivations for each of the following arguments.

a. $\sim(P \leftrightarrow Q)$ <use nb>
$R \vee P$
$\sim Q \rightarrow R$
$\therefore$ R

b. $W \rightarrow U$ <use cdj>
$\sim W \rightarrow V$
$\therefore$ $U \vee V$

c. $P \vee (Q \wedge S)$
$R \vee Q$
$S \vee \sim P$
$Q \rightarrow \sim S$
$\therefore$ R

10 TRUTH TABLES AND TAUTOLOGIES

The sentence 'P ∨ ~P' is logically true. It is true in all logically possible situations. This can be established by simple reasoning:

> Although there are an infinite number of logically possible situations, they fall into two classes. One class consists of situations in which 'P' is true, and the other consists of situations in which 'P' isn't true. In any situation in the first class, 'P ∨ ~P' is true because it is a disjunction whose first disjunct is true. In any situation in the second class, 'P' is not true in that situation, and so its negation, '~P' is true, and again 'P ∨ ~P' is true because it is a disjunction which has a true disjunct. So in either class of situations 'P ∨ ~P' is true.

This pattern of reasoning can be summed up using a **truth table**. The table begins with listing the two options for the truth value of 'P' in a class of situations:

	P	~P	
situations in which 'P' is true →	T		
situations in which 'P' is false →	F		

The truth value of '~P' is determined in each class:

	P	~P	
situations in which 'P' is true →	T	F	
situations in which 'P' is false →	F	T	

and that information determines the truth value of 'P ∨ ~P' in each class:

	P	~P	P ∨ ~P
situations in which 'P' is true →	T	F	T
situations in which 'P' is false →	F	T	T

This is an example in which no matter what truth value the simple parts of the sentence have, the sentence itself is true. Such a sentence is called a "tautology":

> **Tautology**: A sentence is a tautology if no matter what truth values are assigned to its simple parts, the definitions of the connectives used in the sentence determine that the sentence is true.

The truth table just given shows that 'P ∨ ~P' is a tautology, because it shows that 'P ∨ ~P' is true no matter how truth values are assigned to its atomic parts.

This use of truth tables can be applied to a sentence of any degree of complexity. Here is an example showing that 'P∧Q → Q' is a tautology. We begin by listing all of the possible combinations of truth values that 'P' and 'Q' might have. There are four of these: the sentences are both true, the first is true and the second false, the first is false and the second true, or they are both false:

P	Q	P ∧ Q	P∧Q → Q
T	T		
T	F		
F	T		
F	F		

This assignment of truth values to 'P' and to 'Q' determines the truth values of 'P ∧ Q' and of 'P∧Q → Q'; for 'P∧Q → Q':

P	Q	P ∧ Q	P∧Q → Q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

In this table there are all T's under 'P∧Q → Q', showing that it is a tautology.

To handle sentences of arbitrary numbers of sentence letters, we need to have a systematic way of representing all of the possible combinations of truth values that the sentence letters can receive. One way of doing this is to list all of the atomic parts of the sentence on the top of the table. Then, underneath the rightmost letter, write alternations of T and F:

P	Q	R	
		T	
		F	
		T	
		F	
		T	
		F	
		T	
		F	

Under the next letter to its left write alterations of TT and FF:

P	Q	R	
	T	T	
	T	F	
	F	T	
	F	F	
	T	T	
	T	F	
	F	T	
	F	F	

Under the next, write alterations of TTTT and FFFF:

P	Q	R	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Do this until the leftmost letter has gone through one whole set of alterations. If there is one sentence letter, only two rows are required. If there are two, the table will contain four rows. If three, then eight. And so on. There are always 2^n rows in the table when there are n sentence letters.

Next, write the sentence to be tested, and underneath it write in each row the truth value that it has when its parts have the truth values appearing on that row. An example is:

P	Q	R	$P \wedge Q \rightarrow P \wedge R$	
T	T	T	T	
T	T	F	**F**	← not a tautology
T	F	T	T	
T	F	F	T	
F	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	T	

Since there is a row (the second row) in which '$P \wedge Q \rightarrow P \wedge R$' does not have a T, that sentence is not a tautology.

If there is a T in every row, it is a tautology, as in this case:

P	Q	R	$P \wedge Q \rightarrow P \vee R$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

This method is completely mechanical and it always yields an answer in a finite amount of time.

If a sentence is a tautology, then you need to fill in every one of its rows to show that every one is T. But if a sentence is not a tautology, you need only find one row in which the sentence comes out F. For example, the following partial truth table shows that '$P \vee Q \leftrightarrow R \vee Q$' is not a tautology:

P	Q	R	$P \vee Q \leftrightarrow R \vee Q$
T	T	T	
T	T	F	
T	F	T	
T	F	F	**F**
F	T	T	
F	T	F	
F	F	T	
F	F	F	

If a sentence is not a tautology and if you can identify what assignment of truth values to the simple parts will show this, you needn't set up a whole truth table. Just give a single row:

P	Q	R	$P \vee Q \leftrightarrow R \vee Q$
T	F	F	F

and state that this assignment of truth values to sentence letters makes the sentence false.

It turns out that every theorem of the first two chapters of this text is a tautology. This is because the rules and techniques in these chapters only allow derivations of tautologies when no premises are used. It is also a fact that any tautology can be derived by the rules we have. So we have two ways to show tautologies: theorems and truth tables, and we have one way to show non-tautologies: truth tables.

EXERCISES

1. Use truth tables or truth value assignments to determine whether each of these is a tautology.

   a. $(R \leftrightarrow S) \vee (R \leftrightarrow \sim S)$
   b. $R \leftrightarrow (S \leftrightarrow R)$
   c. $R \vee (S \wedge T) \rightarrow R \wedge (S \vee T)$
   d. $\sim U \rightarrow (U \rightarrow \sim V)$
   e. $(\sim R \leftrightarrow R) \rightarrow S$
   f. $(S \wedge T) \vee (S \wedge \sim T) \vee \sim S$

11 TAUTOLOGICAL IMPLICATION

It is easy to show by doing a derivation that this argument is valid:

$P \wedge P$
$\therefore$ P

It is also possible to show that the argument is valid using a technique like that of truth tables. Just show that there is no logically possible situation in which the premise is true and the conclusion false. This can be done as follows:

> All logically possible situations can be divided into two classes. In one class of situations, 'P' is true; no situation of this sort can be one in which the argument has true premises and a false conclusion, because in any of these situations the conclusion, 'P', is true. In all other situations, 'P' is false. But then so is '$P \wedge P$'. So none of these are situations in which the argument has true premises and a false conclusion. So it is valid.

Generalizing, we can say that an argument is valid whenever the premises "tautologically imply" the conclusion. Tautological implication is defined as follows:

> A set of sentences **tautologically implies** a given sentence if and only if there is no assignment of truth values to the atomic parts which make the sentences in the set all true and the given sentence false.

There is a mechanical way to test a set of sentences to see if they tautologically imply a given sentence. Just create a truth table in which all of the sentences in the set, along with the given sentence, each appear at the top of some column. If there is no row in which all of the sentences in the set have T's under them and the given sentence has an F, then that set tautologically implies the given sentence. If there is such a row, then that set does not tautologically imply the given sentence.

Suppose that we are wondering whether this set of sentences: $\{P \to \sim Q,\ R \leftrightarrow P \wedge Q,\ Q \vee R\}$ tautologically implies the sentence $Q \wedge R$. Here is a truth table to test this:

P	Q	R	$P \to \sim Q$	$R \leftrightarrow P \wedge Q$	$Q \vee R$	$Q \wedge R$	
T	T	T	F	T	T	T	
T	T	F	F	F	T	F	
T	F	T	**T**	**T**	**T**	**F**	←
T	F	F	T	T	F	F	
F	T	T	T	F	T	T	
F	T	F	**T**	**T**	**T**	**F**	←
F	F	T	T	F	T	F	
F	F	F	T	T	F	F	

There are in fact two rows (the third and the sixth) in which the sentences in the set are all true and the given sentence is false. So they do not tautologically imply '$Q \wedge R$'.

Using the same technique, we can show that this set of sentences: $\{P \to \sim Q,\ R \leftrightarrow P \wedge Q,\ Q \vee R\}$ does tautologically imply the sentence $P \leftrightarrow R$. The truth table is:

P	Q	R	$P \to \sim Q$	$R \leftrightarrow P \wedge Q$	$Q \vee R$	$P \leftrightarrow R$
T	T	T	F	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	F	T

Here, there is no row in which the sentences in the set are all true and '$P \leftrightarrow R$' is false. So that set of premises does tautologically imply that sentence.

If there is a derivation using the rules and techniques of Chapters 1 and 2 showing that an argument is valid, then the premises of the argument do indeed tautologically imply the conclusion. And vice versa: if some premises tautologically imply a given conclusion, then there is a derivation to show that the argument is valid.

So we have two ways to show that some sentences tautologically imply another: this can be done with a truth table or with a derivation. We have one way to show that some set of sentences does not tautologically imply another: find a way to assign truth values to the simple parts so that all the sentences in the set are all true and the given sentence false.

EXERCISES

For each of the following arguments, either show that the premises tautologically imply the conclusion, or show that the premises do not tautologically imply the conclusion.

a. $U \wedge V \rightarrow X$
$\sim V \rightarrow U$
$X \vee V \rightarrow U$
$\therefore\ V \rightarrow \sim U$

b. $(X \rightarrow Y) \rightarrow Z$
$\sim Z$
$\therefore\ \sim Y$

c. $\sim(P \leftrightarrow Q)$
$R \vee P$
$\sim Q \rightarrow R$
$\therefore\ R$

d. $S \vee T$
$W \vee S$
$\sim T \vee \sim S$
$\therefore\ \sim S$

e. $W \rightarrow U$
$\sim W \rightarrow V$
$\therefore\ U \vee V$

f. $P \leftrightarrow \sim Q$
$Q \rightarrow R \vee P$
$R \rightarrow \sim Q \vee \sim P$
$\therefore\ Q \vee R$

g. $P \vee (Q \wedge S)$
$S \vee Q$
$S \vee \sim P$
$\therefore\ S$

h. $P \wedge (Q \vee S)$
$S \vee Q$
$S \vee P$
$\therefore\ S$

BASIC AND DERIVED RULES FROM CHAPTERS 1 AND 2

BASIC RULES OF CHAPTER 1

Repetition:
□
∴ □

Modus Ponens:
□→○
□
∴ ○

Modus Tollens:
□→○
~○
∴ ~□

Double negation:
□ or ~~□
∴ ~~□ ∴ □

BASIC RULES OF CHAPTER 2

Conjunction rules:

Rule s (simplification)

□ ∧ ○ or □ ∧ ○
∴ □ ∴ ○

Rule adj (adjunction)

□
○
∴ □ ∧ ○

Disjunction rules:

Rule add (addition)

□ or □
∴ □ ∨ ○ ∴ ○ ∨ □

Rule mtp (modus tollendo ponens)

□ ∨ ○ or □ ∨ ○
~○ ~□
∴ □ ∴ ○

Biconditional rules:

Rule bc (biconditional-to-conditional)

□ ↔ ○ or □ ↔ ○
∴ □ → ○ ∴ ○ → □

Rule cb (conditionals-to-biconditional)

□ → ○
○ → □
∴ □ ↔ ○

THEOREMS USED AS RULES

A theorem of conditional form:

$$\therefore \quad \Box \wedge \bigcirc \rightarrow \triangle$$

justifies a rule of this form:

$$\begin{array}{ll} & \Box \\ & \bigcirc \\ \therefore & \triangle \end{array}$$

A theorem of biconditional form:

$$\therefore \; \Box \leftrightarrow \bigcirc$$

justifies whatever both conditionals justify.

DERIVED RULES (May be used if the theorems on which they are based have been derived.)

Rule nc <based on T40>

$$\begin{array}{llcll} & \sim(\Box \rightarrow \bigcirc) & \qquad & & \Box \wedge \sim\bigcirc \\ \therefore & \Box \wedge \sim\bigcirc & & \therefore & \sim(\Box \rightarrow \bigcirc) \end{array}$$

Rule cdj <based on T45, T46>

$$\begin{array}{llcllcllcll} & \Box \rightarrow \bigcirc & \qquad & & \sim\Box \vee \bigcirc & \qquad & & \sim\Box \rightarrow \bigcirc & \qquad & & \Box \vee \bigcirc \\ \therefore & \sim\Box \vee \bigcirc & & \therefore & \Box \rightarrow \bigcirc & & \therefore & \Box \vee \bigcirc & & \therefore & \sim\Box \rightarrow \bigcirc \end{array}$$

Rule sc <based on T33, T49>

$$\begin{array}{llcll} & \Box \vee \bigcirc & \qquad & & \Box \rightarrow \triangle \\ & \Box \rightarrow \triangle & & & \sim\Box \rightarrow \triangle \\ & \bigcirc \rightarrow \triangle & & \therefore & \triangle \\ \therefore & \triangle & & & \end{array}$$

Rule dm <based on T63-66>

$$\begin{array}{llcllcllcll} & \Box \wedge \bigcirc & \qquad & & \Box \vee \bigcirc & \qquad & & \sim(\Box \wedge \bigcirc) & \qquad & & \sim(\Box \vee \bigcirc) \\ \therefore & \sim(\sim\Box \vee \sim\bigcirc) & & \therefore & \sim(\sim\Box \wedge \sim\bigcirc) & & \therefore & \sim\Box \vee \sim\bigcirc & & \therefore & \sim\Box \wedge \sim\bigcirc \end{array}$$

$$\begin{array}{llcllcllcll} & \sim(\sim\Box \vee \sim\bigcirc) & \qquad & & \sim(\sim\Box \wedge \sim\bigcirc) & \qquad & & \sim\Box \vee \sim\bigcirc & \qquad & & \sim\Box \wedge \sim\bigcirc \\ \therefore & \Box \wedge \bigcirc & & \therefore & \Box \vee \bigcirc & & \therefore & \sim(\Box \wedge \bigcirc) & & \therefore & \sim(\Box \vee \bigcirc) \end{array}$$

Rule nb <based on T90>

$$\begin{array}{llcll} & \sim(\Box \leftrightarrow \bigcirc) & \qquad & & \Box \leftrightarrow \sim\bigcirc \\ \therefore & \Box \leftrightarrow \sim\bigcirc & & \therefore & \sim(\Box \leftrightarrow \bigcirc) \end{array}$$

UNABBREVIATED DERIVATIONS

A **derivation** from a set of sentences P consists of a sequence of lines that is built up in order, step by step, where each step is in accordance with these provisions:

- **Show line:** A show line consists of the word "Show" followed by a symbolic sentence. A show line may be introduced at any step. Show lines are not given a justification.
- **Premise**: At any step, any symbolic sentence from the set P may be introduced, justified with the notation "pr".
- **Theorem**: At any step, an instance of a previously proved theorem may be entered with the name of the theorem given as justification. (e.g. "T32")
- **Rule:** At any step, a line may be introduced if it follows by a rule from sentences on previous available lines; it is justified by citing the numbers of those previous lines and the name of the rule. This includes the following basic rules:

r	mp	mt	dn	s
adj	add	mtp	bc	cb

  It also includes rules based on previously derived theorems, where the name of a rule based on a theorem is "R" followed by the name of the theorem; e.g. "RT32". If the appropriate enabling theorems have been derived, these rules are also available for use:

nc	cdj	sc	dm	nb
- **Direct derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the sentence on the closest previous uncancelled show line, one may, as the next step, write "dd" following the justification for that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for conditional derivation**: When a show line with a conditional sentence is introduced, as the next step one may introduce an immediately following line with the antecedent of the conditional on it; the justification is "ass cd".
- **Conditional derivation:** When a line (which is not a show line) is introduced whose sentence is the same as the consequent of the conditional sentence on the closest previous uncancelled show line, one may, as the next step, write "cd" at the end of that line, draw a line through the word "Show", and draw a box around all the lines below the show line, including the current line.
- **Assumption for indirect derivation**: When a show line is introduced, as the next step one may introduce an immediately following line with the [un]negation of the sentence on the show line; the justification is "ass id".
- **Indirect derivation:** When a sentence is introduced on a line which is not a show line, if there is a previous available line containing the [un]negation of that sentence, and if there is no uncancelled show line between the two sentences, as the next step you may write the line number of the first sentence followed by "id" at the end of the line with the second sentence. Then you cancel the closest previous "show", and box all sentences below that show line, including the current line.

Except for steps that involve boxing and canceling, every step introduces a line. When writing out a derivation, every line that is introduced is written directly below previously introduced lines.

Optional variant: When boxing and canceling with direct or conditional derivation, the "dd" or "cd" justification may be written on a later line which contains no sentence at all, and which is followed by the number of the line that satisfies the conditions for direct or conditional derivation. With indirect derivation, the "id" justification may be written on a later line which contains no sentence at all, and which is followed by the numbers of the two lines containing contradictory sentences. In all cases, the lines cited must be available from the later line.

STRATEGY HINTS

Try to reason out the argument for yourself.

Begin with a sketch of an outline of a derivation, and then fill in the details.

Write down obvious consequences.

When no other strategy is obvious, try indirect derivation.

To derive:	**Try this:**
Conjunction □ ∧ ○	Derive each conjunct, and adjoin them
Disjunction □ ∨ ○	Derive either disjunct and use add. (Often this is not possible.) Assume '~(□ ∨ ○)' for id and immediately use dm. Derive '~□ → ○' and use cdj
Conditional □ → ○	Use cd
Biconditional □ ↔ ○	Derive both conditionals and use cb.
Negation of conjunction **~(□ ∧ ○)**	Use id.
Negation of disjunction **~(□ ∨ ○)**	Derive '~□ ∧ ~○' and use dm. Perhaps assume '□ ∨ ○' for id and try to derive both '□ → P∧~P' and '○→P∧~P'. Then use sc (applied to the assumed '□ ∨ ○' and the conditionals) to derive 'P∧~P'.
Negation of conditional **~(□ → ○)**	Use id.
Negation of biconditional **~(□ ↔ ○)**	Derive '□ ↔ ~○' and use nb.

If you have this available:	**Try this:**
Conjunction □ ∧ ○	Simplify and use the conjuncts singly.
Disjunction □ ∨ ○	Try to derive the negation of one of the disjuncts, and use mtp.. Derive the conditionals '□ → △' and '○ → △', where '△' is something you want to derive. Then use sc with the disjunction and two conditionals.
Conditional □ → ○	Try to derive the antecedent to set up mp, or derive the negation of the consequent, to set up mt.
Biconditional □ ↔ ○	Infer both conditionals and use them with mp, mt, and so on.
Negation of	Use dm to turn this into '~□ ∨ ~○', and then try to derive either '□' or '○' to use mtp.

conjunction **~(□ ∧ ○)**	
Negation of disjunction ~(□ ∨ ○)	Use dm to turn this into '~□ ∧ ~○'; then simplify and use the conjuncts singly.
Negation of conditional ~(□ → ○)	Use nc to derive '□ ∧ ~○', then simplify and use the conjuncts singly.
Negation of biconditional ~(□ ↔ ○)	Use nb to turn this into '□ ↔ ~○', and use bc to get the corresponding conditionals.

ALL THEOREMS FROM CHAPTERS 1 AND 2

T1	**P→P**	
T2	**Q → (P→Q)**	
T3	**P → ((P→Q) → Q)**	
T4	**(P→Q) → ((Q→R) → (P→R))**	Syllogism
T5	**(Q→R) → ((P→Q) → (P→R))**	Syllogism
T6	**(P → (Q→R)) → ((P→Q) → (P→R))**	Distribution of → over →
T7	**((P→Q) → (P→R)) → (P→(Q → R))**	Distribution of → over →
T8	**(P→ (Q→R)) → (Q → (P→R))**	Commutation
T9	**(P → (P→Q)) → (P→Q)**	
T10	**((P→Q) → Q) → ((Q→P) → P)**	
T11	**~~P → P**	Double negation
T12	**P → ~~P**	Double negation
T13	**(P→Q) → (~Q → ~P)**	Transposition
T14	**(P → ~Q) → (Q → ~P)**	Transposition
T15	**(~P → Q) → (~Q → P)**	Transposition
T16	**(~P → ~Q) → (Q → P)**	Transposition
T17	**P → (~P→Q)**	
T18	**~P → (P→Q)**	
T19	**(~P→P) → P**	Reductio ad absurdum
T20	**(P→~P) → ~P**	Reductio ad absurdum
T21	**~(P→Q) → P**	
T22	**~(P→Q) → ~Q**	
T23	**((P→Q) → P) → P**	Peirce's law
T24	**P∧Q ↔ Q∧P**	"commutative law for conjunction"

T25	$P\wedge (Q\wedge R) \leftrightarrow (P\wedge Q)\wedge R$	"associative law for conjunction"
T26	$(P\to Q) \wedge(Q\to R) \to (P\to R)$	"hypothetical syllogism"
T27	$(P\wedge Q \to R) \leftrightarrow (P \to (Q\to R))$	"exportation"
T28	$(P\wedge Q \to R) \leftrightarrow (P\wedge \sim R \to \sim Q)$	
T29	$(P\to Q\wedge R) \leftrightarrow (P\to Q) \wedge(P\to R)$	"distribution of $\to$ over $\wedge$"
T30	$(P\to Q) \to (R\wedge P \to R\wedge Q)$	
T31	$(P\to Q) \to (P\wedge R \to Q\wedge R)$	
T32	$(P\to R) \wedge(Q\to S) \to (P\wedge Q \to R\wedge S)$	"Leibniz's *praeclarum theorema*"
T33	$(P\to Q) \wedge(\sim P\to Q) \to Q$	"separation of cases; constructive dilemma"
T34	$(P\to Q) \wedge(P\to \sim Q) \to \sim P$	"reductio ad absurdum"
T35	$(\sim P\to R) \wedge(Q\to R) \leftrightarrow ((P\to Q) \to R)$	
T36	$\sim(P \wedge \sim P)$	"non-contradiction"
T37	$(P\to Q) \leftrightarrow \sim(P \wedge \sim Q)$	
T38	$P\wedge Q \leftrightarrow \sim(P \to \sim Q)$	
T39	$\sim(P\wedge Q) \leftrightarrow (P \to \sim Q)$	
T40	$\sim(P \to Q) \leftrightarrow P\wedge \sim Q$	"negation of conditional"
T41	$P \leftrightarrow P\wedge P$	"idempotence for $\wedge$"
T42	$P\wedge \sim Q \to \sim(P\to Q)$	"negation of conditional"
T43	$\sim P \to \sim(P\wedge Q)$	
T44	$\sim Q \to \sim(P\wedge Q)$	
T45	$P\vee Q \leftrightarrow (\sim P\to Q)$	
T46	$(P\to Q) \leftrightarrow \sim P \vee Q$	"definition of $\to$ in terms of $\vee$"
T47	$P \leftrightarrow P\vee P$	"idempotence for $\vee$"
T48	$(P\vee Q) \wedge(P\to R) \wedge(Q\to S) \to R\vee S$	
T49	$(P\vee Q) \wedge(P\to R) \wedge(Q\to R) \to R$	"separation of cases"
T50	$(P\to R) \wedge(Q\to R) \leftrightarrow (P\vee Q\to R)$	
T51	$(P\vee Q) \wedge(P\to R) \wedge(\sim P\wedge Q\to R) \to R$	
T52	$(P\to R) \wedge(\sim P\wedge Q\to R) \leftrightarrow (P\vee Q\to R)$	
T53	$P\vee Q \leftrightarrow Q\vee P$	"commutative law for $\vee$"
T54	$P \vee (Q\vee R) \leftrightarrow (P\vee QR) \vee R$	"associative law for $\vee$"
T55	$(P \to Q\vee R) \leftrightarrow (P\to Q) \vee (P\to R)$	"distribution of $\to$ over $\vee$"
T56	$(P\to Q) \to (R\vee P \to R\vee Q)$	
T57	$(P\to Q) \to (P\vee R \to Q\vee R)$	
T58	$(P\to Q) \vee (Q\to R)$	
T59	$P \vee \sim P$	"excluded middle"
T60	$(P\to R) \vee (Q\to R) \leftrightarrow (P\wedge Q \to R)$	

T61	$P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$	"distribution"
T62	$P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$	"distribution"
T63	$P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$	"de morgans"
T64	$P \vee Q \leftrightarrow \sim(\sim P \wedge \sim Q)$	"de morgans"
T65	$\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$	"de morgans"
T66	$\sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$	"de morgans"
T67	$\sim P \wedge \sim Q \rightarrow \sim(P \vee Q)$	
T68	$P \leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q)$	
T69	$P \leftrightarrow (P \vee Q) \wedge (P \vee \sim Q)$	
T70	$Q \rightarrow (P \wedge Q \leftrightarrow P)$	
T71	$\sim Q \rightarrow (P \vee Q \leftrightarrow P)$	
T72	$(P \rightarrow Q) \leftrightarrow (P \wedge Q \leftrightarrow P)$	
T73	$(P \rightarrow Q) \leftrightarrow (P \vee Q \leftrightarrow Q)$	
T74	$(P \leftrightarrow Q) \wedge P \rightarrow Q$	
T75	$(P \leftrightarrow Q) \wedge Q \rightarrow P$	
T76	$(P \leftrightarrow Q) \wedge \sim P \rightarrow \sim Q$	
T77	$(P \leftrightarrow Q) \wedge \sim Q \rightarrow \sim P$	
T78	$(P \rightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \rightarrow Q) \leftrightarrow (P \rightarrow R))$	
T79	$(P \rightarrow (Q \leftrightarrow R)) \leftrightarrow (P \wedge Q \leftrightarrow P \wedge R)$	
T80	$(P \leftrightarrow Q) \vee (P \leftrightarrow \sim Q)$	
T81	$(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	
T82	$(P \leftrightarrow Q) \leftrightarrow \sim((P \rightarrow Q) \rightarrow \sim(Q \rightarrow P))$	
T83	$(P \leftrightarrow Q) \leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$	
T84	$P \wedge Q \rightarrow (P \leftrightarrow Q)$	
T86	$((P \leftrightarrow Q) \rightarrow R) \leftrightarrow (P \wedge Q \rightarrow R) \wedge (\sim P \wedge \sim Q \rightarrow R)$	
T87	$\sim(P \leftrightarrow Q) \leftrightarrow (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$	
T88	$P \wedge \sim Q \rightarrow \sim(P \leftrightarrow Q)$	
T89	$\sim P \wedge Q \rightarrow \sim(P \leftrightarrow Q)$	
T90	$\sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$	
T91	$P \leftrightarrow P$	
T92	$(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$	
T93	$(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \rightarrow (P \leftrightarrow R)$	
T94	$(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$	
T95	$(P \rightarrow Q) \leftrightarrow ((P \leftrightarrow R) \leftrightarrow (Q \leftrightarrow R))$	
T96	$(P \leftrightarrow Q) \leftrightarrow (\sim P \leftrightarrow \sim Q)$	

T97 $(P\leftrightarrow R)\wedge (Q\leftrightarrow S) \to ((P\to Q) \leftrightarrow (R\to S))$

T98 $(P\leftrightarrow R)\wedge (Q\leftrightarrow S) \to (P\wedge Q \leftrightarrow R\wedge S)$

T99 $(P\leftrightarrow R)\wedge (Q\leftrightarrow S) \to (P\vee R \leftrightarrow Q\vee S)$

T100 $(P\leftrightarrow R)\wedge (Q\leftrightarrow S) \to ((P\leftrightarrow Q) \leftrightarrow (R\leftrightarrow S))$

T101 $(Q\leftrightarrow S) \to ((P\to Q) \leftrightarrow (P\to S)) \wedge((Q\to P) \leftrightarrow (S\to P))$

T102 $(Q\leftrightarrow S) \to (P\wedge Q \leftrightarrow P\wedge S)$

T103 $(Q\leftrightarrow S) \to (P\vee Q \leftrightarrow P\vee S)$

T104 $(Q\leftrightarrow S) \to ((P\leftrightarrow Q) \leftrightarrow (P\leftrightarrow S))$

T105 $P\wedge (Q\leftrightarrow R) \to (P\wedge Q \leftrightarrow R)$

T106 $(P \to (Q\to R)) \leftrightarrow ((P\to Q) \to (P\to R))$

T107 $(P \to\to R)) \leftrightarrow (Q \to (P\to R))$

T108 $(P \to(P\to Q)) \leftrightarrow(P\to Q)$

T109 $((P\to Q) \to Q) \leftrightarrow ((Q\to P) \to P)$

T110 $P \leftrightarrow {\sim}{\sim}P$

T111 $(P\to Q) \leftrightarrow ({\sim}Q\to {\sim}P)$

T112 $(P\to {\sim}Q) \leftrightarrow (Q\to {\sim}P)$

T113 $({\sim}P\to Q) \leftrightarrow ({\sim}Q\to P)$

T114 $({\sim}P\to P) \leftrightarrow P$

T115 $(P\to {\sim}P) \leftrightarrow {\sim}P$

T116 $(P\wedge Q) \vee (R\wedge S) \leftrightarrow (P\vee R) \wedge(P\vee S) \wedge(Q\vee R) \wedge(Q\vee S)$

T117 $(P\vee Q) \wedge(R\vee S) \leftrightarrow (P\wedge R) \vee (P\wedge S) \vee (Q\wedge R) \vee (Q\wedge S)$

T118 $(P\vee Q) \wedge(R\vee S) \leftrightarrow ({\sim}P\wedge {\sim}R) \vee ({\sim}P\wedge S) \vee (Q\wedge {\sim}R) \vee (Q\wedge S)$

T119 $(P\vee {\sim}P) \wedge Q \leftrightarrow Q$

T120 $(P\wedge {\sim}P) \vee Q \leftrightarrow Q$

T121 $P \vee ({\sim}P\wedge Q) \leftrightarrow P\vee Q$

T122 $P \wedge ({\sim}P\vee Q) \leftrightarrow P\wedge Q$

T123 $P \leftrightarrow P \vee (P\wedge Q)$

T124 $P \leftrightarrow P \wedge (P\vee Q)$

T125 $(P\to Q\wedge R) \to (P\wedge Q \leftrightarrow P\wedge R)$