Chapter Three
Name letters, Predicates, Variables and Quantifiers

1 NAME LETTERS AND PREDICATES

In chapters 1 and 2 we studied logical relations that depend only on the sentential connectives: '~', '→', '∧', '∨', '↔'. The atomic sentences -- those that contain no connectives -- were symbolized by sentential letters, and we paid no attention to any internal structure that they might have. It is now time to study that structure. The Predicate Calculus is a system of logic that studies the ways in which sentences are constructed out of name letters, predicates, variables, and quantifiers, as well as connectives. We have already studied connectives; in this section we introduce name letters, predicates, variables, and quantifiers.

In our logical symbolism, **name letters** are written as small letters: a, b, c, d, e, f, g, h. Any small letter between 'a' and 'h' can be used as a name letter. Name letters in the logical symbolism correspond to names of English:

> *Carlos, Agatha, Dr. Samuelson, Ms. Bernstein, Madame Curie, David Rockefeller, San Diego, Germany, UCLA, General Electric, Microsoft, Google, Macy's, The Los Angeles Times, I-405, Memorial Day, the FBI, ...*

Any one of these may be symbolized by means of a name letter:

h *Henry*
c *California*
g General Electric

The simplest way to make a sentence containing a name letter is to combine it with a **one-place predicate**. One-place predicates appear in our logical symbolism as the capital letters from A to O. One-place predicates correspond roughly to grammatical predicates in English; in the following examples, the underlined phrases would be symbolized as one-place predicates:

Agatha <u>is clever</u>.
Henry <u>is a giraffe</u>.
Ferdy <u>dances well</u>.
Georgia <u>is a state</u>
Ann <u>will run for re-election</u>.

(The parts that are not underlined are symbolized with name letters.)

Whereas English proper names are usually capitalized, the logical name letters that represent them are not, and whereas English predicates are typically not capitalized, the logical predicates that represent them are capitalized. There is nothing "logical" about this reverse convention; it is an historical accident, but it has now become part of the tradition of symbolic logic Further, in the usual formulations of the predicate calculus the predicate comes before the name letter, instead of after it as in English. This, too, is an historical accident. So the sentences given above can be symbolized as follows:

Agatha is clever.	Ca
Henry is a giraffe.	Gh
Ferdy *dances well*.	Df
Georgia is a state.	Ag
Ann *will run for re-election*.	Ea

> A **one-place** ("monadic") predicate is any capital letter between 'A' and 'O'.
> A name **letter** is any small letter from 'a' to 'h'.
> An atomic sentence may be formed by writing a one-place predicate followed by a name letter.

EXERCISES

1. Symbolize each of the following sentences:

a. Fred is an orangutan.
b. Gertrude is an orangutan but Fred isn't.
c. Tony Blair will speak first.
d. Gary lost weight recently; he is happy.
e. Felix cleaned and polished.
f. Darlene or Abe will bat clean-up.

We assume that a one-place predicate is true of certain things, and that a name letter stands for a unique thing. A sentence consisting of a one-place predicate together with a name letter is true if and only if the predicate is true of the thing that the name letter stands for. Thus, taking the examples listed above, we assume that 'C' is true of all and only clever things, that 'a' stands for Agatha (presumably a person or animal), and then:

Ca

is true if and only if Agatha is one of the clever things that the predicate is true of. Similarly, if `G' is true of giraffes, then `Gh' is true if Henry is one of the giraffes. If `E' is true of the things that will run for re-election, and if 'a' stands for Ann, then `Ea' is true if and only if Ann will run for reelection.

Predicates are generally true of several specific things, but a predicate might be true of only one thing ('is a moon of the earth') or might not be true of anything at all. If there are in fact no dragons, the sentence:

Df *Fred is a dragon*

contains a predicate 'D' that is true of nothing at all. This means that the sentence `Df' will be false, no matter who or what `Fred' stands for.

In this chapter we assume that each name letters in our logical symbolism stands for a unique thing. This assumption is an idealization, for it is not true that the words of English that we are representing by name letters always succeed in naming something. If there is no such person as Paul Bunyan, then `Paul Bunyan' is a "name" that names nothing at all. In some systems of logic it is possible to use name letters which do not stand for anything; these systems of logic are called "free logics". (They are called "free" because they are "free of" the assumption that the name letters they contain actually stand for things.) Free logics are a bit more complicated than standard logic. (Studies of free logic assume that the reader is already acquainted with the standard logic taught here.) In this text we assume that any name letter that we use stands for something.

EXERCISES

2. Symbolize each of the following, assuming:

`D' is true of doctors
'L' is true of people who are in love
'h' stands for Hans
'a' stands for Amanda

a. *Hans is a doctor but Amanda isn't.*
b. *Hans, who is a doctor, is in love*
c. *Hans is in love but Amanda isn't*
d. *Neither Hans nor Amanda is in love*
f. *Hans and Amanda are both doctors*.

3. Symbolize each of the following, using:

'L' for things that live in Brea
'D' for things that drive to school

a. Eileen *and Cosi both live in Brea*.
b. Eileen *drives to school, and so does Hank*.
c. *If Hank lives in Brea then he drives to school; otherwise he doesn't drive to school.*
d. *If David and Hank both live in Brea then David drives to school but Hank doesn't.*
e. *Neither Hank nor Eileen live in Brea, yet each of them drives to school.*

2 QUANTIFIERS, VARIABLES, AND FORMULAS

So far, we have no means at all in our symbolism to express generalities. We can say that Pedro is a doctor, and we can say that Pedro is wealthy, but we cannot say that everyone is a doctor, or that every doctor is wealthy. Nor can we deny that everyone is a doctor, or say that some doctor isn't wealthy. We cannot even express these claims. In order to express generalities we will introduce **quantifiers** and **variables**.

Variables: Any small letter from 'i' to 'z' is a variable. Small letters between 'l' and 'z' with numerical subscripts are also variables (though they will not be used in this chapter).

The **universal quantifier** is '$\forall$'.
The **existential quantifier** is '$\exists$'.

A quantifier phrase is a quantifier followed by a variable:
$\forall x \qquad \forall z \qquad \forall s \qquad \exists x \qquad \exists z \qquad \exists s$

Here is how we use quantifiers. Suppose that we wish to say -- as some philosophers have said -- that everything in the universe is either mental or physical. Suppose that `M' is the one-place predicate `*is mental*', and `H' is the one-place predicate `*is physical*'. Then we symbolize the claim that *everything is either mental or physical* as follows:

$\forall x(Mx \vee Hx)$.

The initial `$\forall x$' is a universal quantifier phrase. This is followed by something, `$(Mx \vee Hx)$', which we will call a **symbolic formula**. A formula is something like a symbolic sentence, except that in addition to a name letter following each predicate we may instead have a variable, such as `x' above. The displayed formula says that everything satisfies a certain condition. The universal quantifier phrase is responsible for the "everything" part, and the combination of variables and predicates tells us what the condition is. In the case in point, the condition is that it is either mental or physical:

$$\underbrace{\forall x}_{\textit{Everything is such that}}\ \underbrace{(Mx \vee Hx)}_{\textit{it is either mental or physical}}$$

The existential quantifier can appear in a formula in the same place that a universal quantifier may appear:

$$\underbrace{\exists x}_{\textit{Something is such that}}\ \underbrace{(Mx \vee Hx)}_{\textit{it is either mental or physical}}$$

In order to construct sentences in our new extended notation, we begin by defining what a *symbolic formula* is. Intuitively, a symbolic formula is like a sentence, except that it may contain variables in places where name letters otherwise would appear. We use the word 'term' to cover both name letters and variables.

> **Terms**: Any name letter or variable is a term.

So 'a' and 'x' are both terms. A formula is built up in steps, as follows:

> **Sentence letters**: Any sentence letter is an atomic formula.

> **Atomic formulas:** A one-place predicate followed by a term is an atomic formula.
>
> Thus, if F is a one-place predicate and 'a' is a name letter, then 'Fa' is an atomic formula, and
> if 'F' is a one-place predicate and 'x' is a variable then 'Fx' is an atomic formula.

Both 'Henry is a giraffe' and 'x is a giraffe' are symbolized as atomic formulas:

Gh,   Gx

> **Molecular formulas**: If □ and ○ are formulas, then the following are molecular formulas:
>
> ~□   □∧○   □∨○   □→○   □↔○

Here are some molecular formulas:

~Gh   ~Gx   (Gx ∧ Fa)   (Gx ∨ Jc)   (Gh → Jy)   (~Fa ↔ Ga) → Hx

We can also make formulas out of other formulas by "generalizing" them with quantifiers:

> **Quantified formulas**: If □ is a formula, and 'x' is a variable, then these are quantified formulas:
>
> ∀x□   ∃x□

Examples of quantified formulas are:

∀xGx   ∃xFx   ∀y(Gy→Fy)   ∃w~(Gw ∧ ~Fb)   ∀v(~Jx ↔ Fv)

Once a quantified formula is constructed, it may be used as input to any of these provisions. So, given that the examples above are formulas, we can make new formulas by combining them with connectives:

(∀xGx ∧ ∃xFx)   (∃xFx ∨ ∀y(Gy→Fy))   ∀y(Gy→Fy)   (P ∧ ∃xFx)

We may informally omit parentheses exactly as we did in the last chapter, to produce informal notation:

∀xGx ∧ ∃xFx   ∃xFx ∨ ∀y(Gy→Fy)

(Note that '∀yGy→Fy', is a conditional; it is not equivalent to '∀y(Gy→Fy)', which is a universal generalization of a conditional.)

Likewise, we can add a quantifier to a formula that already has one or several quantifiers within it:

∀x(Gx → ∃yFy)   ∀x~∃y(Gx ∨ ~Fy)   ∀x∀y∀z(Gx → Fz)

> A **formula** is anything that can be constructed by means of the above provisions for atomic formulas, molecular formulas, and quantified formulas.
> Nothing else is a formula.

Every formula is either atomic, or it has a main connective or a quantifier with scope over the whole formula. The main connective or quantifier in a formula is the last connective or quantifier that was added in constructing the formula. Formulas may be parsed as in chapters 1 or 2. Some examples are:

```
∀x(Gx → ∃yFy)                        ∀x~∃y(Gx ∨ ~Fy)
 |                                     |
(Gx → ∃yFy)                          ~∃y(Gx ∨ ~Fy)
   /   \                              |
Gx       ∃yFy                          ∃y(Gx ∨ ~Fy)
          |                            |
          Fy                         (Gx ∨ ~Fy)
                                        /   \
                                     Gx      ~Fy
                                              |
                                              Fy
```

EXERCISES

1. For each of the following, say whether it is a formula in official notation, or in informal notation, or not a formula at all. If it is a formula, parse it.

a. ~∀x(Fx → (Gx ∧ Hx))
b. ∃x~~Gx → Hx ∨ ∃yGy
c. ~(Gx ↔ ~Hx)
d. ∀xGx ∧ ∃Hx
e. Fa → (Gb ↔ Hc)
f. ∀x(Gx ↔ x ∨ Ha)
g. ∀x(Gx ↔ Hx) → Ha ∧ ∃zKz

3 SCOPE AND BINDING

In the following we will need to distinguish a symbol from an occurrence of that symbol. For example, the formula:

$\forall xFx$

contains one variable, the variable 'x', which occurs twice in the formula. It has one occurrence immediately following the quantifier, and one occurrence immediately following the predicate 'F'. It will be important to be able to say when an occurrence of a quantifier **binds** an occurrence of a variable. This can be given a precise explanation in terms of the **scope** of an occurrence of a quantifier. The scope of an occurrence of a quantifier includes itself and its variable along with the formula to which it was prefixed when constructing the whole formula. Here are some occurrences of quantifiers and their scopes, indicated by underlining. (The line immediately under a quantifier occurrence indicates its scope.)

$\underline{\forall x\ Fx}$

$\underline{\forall x(Fx \rightarrow Gx)}$

$\underline{\exists xFx} \wedge \underline{\exists y(Gy \wedge Hy)}$

$\underline{\exists x(Fx \wedge \underline{\forall yGy})}$

$\underline{\exists x(Fx \wedge \underline{\exists y(\underline{\exists zGz} \wedge Hy)})}$

Using the notion of the scope of a quantifier, we can say when a quantifier occurrence binds an occurrence of a variable in a formula:

> A quantifier occurrence **binds** an occurrence of a variable if
> the variable occurrence is within the scope of the quantifier occurrence
> the variable occurrence is the same as the one that accompanies the quantifier
> the variable occurrence is not already bound by another quantifier occurrence within the scope of the first quantifier occurrence

(Notice that a variable occurrence that is part of a quantifier phrase is automatically bound by its quantifier.)

The arrows here indicate which variables are bound by the quantifier:

$\forall x(Fx \rightarrow Gx)$

The initial quantifier binds both occurrences of 'x' because (1) they are within its scope, (2) they are the same letter as the one in the quantifier itself, and (3) they are not already bound by another quantifier in the formula. These examples are similar:

$\exists xFx \wedge \exists y(Gy \wedge Hy)$

$\exists x(Fx \wedge \forall yGy)$

$\exists x(Fx \wedge \exists y\ (\exists zGz \wedge Hy))$

$\exists x(Fx \wedge \exists y\ (\exists zGz \wedge Hy \wedge Hx))$

The following example illustrates a case in which an occurrence of 'x' (the last one) is not bound by the initial quantifier'∃x′, even though it is within its scope. This is because there is another quantifier inside that already binds that occurrence of 'x':

$\exists x(Fx \wedge \exists x\ (\exists zGz \wedge Hx))$

Using the notion of a quantifier binding an occurrence of a variable, we can define what a sentence is:

> A **sentence** is any formula in which every occurrence of a variable in the formula is bound by an occurrence of a quantifier in the formula.

A variable occurrence that is not bound is called "free". So a sentence can also be defined as a formula that contains no free occurrences of variables.

All of the examples given above are sentences. The following formulas are not sentences because certain occurrences of variables in them are not bound any of their quantifiers:

$\forall x(Fy \rightarrow Gx)$	no quantifier contains 'y'
$\exists xFx \wedge \exists y(Gx \wedge Hy)$	the scope of the initial quantifier does not include the second 'x'
$\exists x(Fx \wedge \exists y\ (\exists zGz \wedge Hz))$	the scope of the quantifier with 'z' does not extend far enough
$\exists x(Fx \wedge \exists x(\exists zGz \wedge Hy))$	no quantifier contains 'y'

EXERCISES

1. For each of the following, say whether it is a sentence, a formula that is not a sentence, or not a formula at all. (Include sentences and formulas in informal notation as sentences and formulas.) If it is a sentence or formula, indicate which quantifiers bind which variables.

a. $\exists x(Fx \wedge \forall y(Gy \vee Hx))$
b. $\exists y(Hy \wedge \exists{\sim}zHz)$
c. $\exists z{\sim}(Hz \wedge Gx \wedge \exists xIx)$
d. ${\sim}({\sim}Gx \rightarrow \forall y(Jx \wedge Ky \leftrightarrow Lx))$
e. $\exists xGx \leftrightarrow \exists y(Gy \wedge Hx)$
f. $\forall x(Gx \rightarrow \forall y(Hy \rightarrow \forall z(Iz \rightarrow Hx \wedge Gz)))$
g. $\forall x\exists y(Hx \leftrightarrow {\sim}Gy)$
h. $\forall xy(Gx \wedge Hy \rightarrow Kx)$
i. $\forall x(Gx \wedge \exists y \rightarrow Hx \wedge Jy)$
j. $\forall x\exists y\forall z(Gx \leftrightarrow \exists w(Hw \wedge {\sim}Hx \wedge Gy))$

4 MEANINGS OF THE QUANTIFIERS

What do quantifiers mean? This can be answered indirectly by giving a way to read symbolic formulas in English. We already know how to read the parts of formulas without quantifiers or variables; we have:

Gh	*Henry is a giraffe*
Ea	Ann *with run for reelection*
Gh $\wedge$ Ea	*Henry is a giraffe and Ann will run for reelection.*
Gh $\rightarrow$ Ea	*If Henry is a giraffe then Ann will run for reelection.*

We can read a quantified formula by adding this:

> Read any universal quantifier as "everything is such that", while reading any variable that it binds as a pronoun which has the 'everything' as its antecedent.
> Read any existential quantifier as "something is such that" while reading any variable that it binds as a pronoun which has the 'something' as its antecedent.

Here are some examples:

$\forall$xGx	*everything is such that it is a giraffe*
$\exists$x(Gx $\wedge$ Ex)	*something is such that it is a giraffe and it will run for reelection*
$\forall$x(Gx $\rightarrow$ Ex)	*everything is such that if it is a giraffe then it will run for reelection*

These readings are stilted, and sometimes cumbersome. But they are accurate paraphrases of the symbolic notation. Often there are more natural ways to word an English sentence. For example, these are all equivalent:

$\exists$x(Gx $\wedge$ Ex)
something is such that it is a giraffe and it will run for reelection
something is a giraffe which will run for reelection
some giraffe will run for reelection

Likewise, these are all equivalent:

$\forall$x(Gx $\rightarrow$ Ex)
everything is such that if it is a giraffe then it will run for reelection
everything, if it is a giraffe, will run for reelection
every giraffe will run for reelection

As in the case of connectives, we need to distinguish carefully between the official definition of the quantifiers and the question of how best to read them in English. The official definition of the quantifiers has to do with the truth-values of the sentences that are produced using them:

> **Definitions of the quantifiers**
>
> To tell whether or not a sentence of the form $\forall$x(...x...x...) is true:
>
> Remove the initial universal quantifier. Pretend that the variable it was binding is a name letter. If you now have a sentence that is true no matter what the pretend constant stands for, then the original sentence is true; otherwise it is false.
>
> To tell whether or not a sentence of the form $\exists$x(...x...x...) is true:
>
> Remove the initial existential quantifier. Pretend that the variable it was binding is a name letter. If there is something that the pretend constant could stand for such that the sentence you now have is true, then the original sentence is true; otherwise it is false.

To apply this to the example '*Everything is either mental or physical*':

> Begin with the sentence:
>
> > $\forall x(Mx \vee Hx)$.
>
> Erase the initial quantifier, yielding:
>
> > $Mx \vee Hx$.
>
> Now pretend that `x' is a name letter, and ask ourselves:
>
> > Is `$Mx \vee Hx$' true no matter what `x' stands for?
>
> If the answer is *yes*, then the original sentence `$\forall x(Mx \vee Hx)$' is true; otherwise `$\forall x(Mx \vee Hx)$' is false.

This test explains why we read `$\forall x(Mx \vee Hx)$' in English as `*Everything is either mental or physical*'. It is because the test for the truth of `$\forall x(Mx \vee Hx)$' succeeds if everything is indeed either mental or physical, and it fails if not everything is either mental or physical. To see that this is so, compare the meaning of the English sentence with the official statement of the conditions under which the symbolized version is true:

> Suppose that certain philosophers are right, and everything *is* either mental or physical. Then if we treat `x' as a name letter, the phrase `$Mx \vee Hx$' must be true no matter what `x' stands for. Because it can only stand for something that is mental or physical (that's all there is), and if it stands for something mental the first disjunct is satisfied, and if it stands for something physical then the second disjunct is satisfied.
>
> Suppose on the other hand that not everything is either mental or physical. (Suppose, as some philosophers have argued, that the number 4 is neither a mental thing nor a physical thing.) Then if we treat `x' as a name letter, we will not find that the phrase `$Mx \vee Hx$' is true no matter what `x' stands for. For if `x' stands for the number 4, neither disjunct will be satisfied.

These considerations do not settle the question of *whether* everything is either mental or physical. Instead they show that there is an equivalence between the truth-value, in English, of the sentence `*Everything is either mental or physical*', and the truth-value, according to our official account, of the predicate calculus sentence `$\forall x(Mx \vee Hx)$'.

EXERCISES

1. Suppose that `A' stands for `is a sofa', `B' stands for `is well-built' and `C' stands for `is comfortable'. For each of the following sentences, produce an accurate but "cumbersome" reading in English as well as a natural idiomatic reading if possible.

a. $\exists x(Ax \wedge Bx)$	e. $\forall y{\sim}Ay$
b. $\forall x(Ax \rightarrow Bx)$	f. $\forall z(Az \wedge Bz \rightarrow Cz)$
c. $\forall x(Ax \vee Bx)$	g. $\exists xCx \wedge \forall yBy$
d. $\exists x{\sim}Ax$	h. $\exists x(Cx \rightarrow \forall yBy)$

2. Assume that all giraffes are friendly, and that some giraffes are clever and some aren't. What are the truth-values of these sentences?

a. $\forall x(Gx \rightarrow Fx)$	d. $\exists y(Fy \wedge Cy)$
b. $\forall x(Gx \rightarrow Cx)$	e. $\exists z(Gz \wedge Cz)$
c. $\exists x({\sim}Fx \wedge Gx)$	f. $\forall x(Gx \rightarrow {\sim}Gx)$

5 SYMBOLIZING SENTENCES WITH QUANTIFIERS

5A CATEGORICAL SENTENCES

The ancient Greek philosopher Aristotle is generally credited with the invention of formal logic. He devised a fairly complete and accurate study of the logical relations among sentences of a certain special sort. These are called "categorical" sentences, and they include any sentence which has one of the following forms (with Aristotle's titles):

Universal affirmative:	*Every A is B*
Particular affirmative:	*Some A is B*
Universal negative:	*No A is B*
Particular negative:	*Some A is not B*

These categorical sentences are only a few of the forms that can be represented in modern predicate logic, but they are simple and basic, and their treatment provides a nice introduction to the symbolism.

A **universal affirmative** sentence of the form:

Every A is B

is represented in the predicate calculus as:

$\forall x(Ax \rightarrow Bx)$.

You can judge the adequacy of this for yourself by comparing the reading of the symbolic version with the English form; that is, compare:

Everything is such that if it is an A then it is B

with:

Every A is B.

The question to ask for logical purposes is: Is there any possible situation in which these two sentences differ in truth-value? If they agree in all logically possible situations, then the proposed symbolization is a good one; otherwise not. Here is some reasoning that suggests the symbolization is a good one:

> Suppose that in some possible situation every A is B. Then, in that situation everything will be such that if it's an A then it is B. Suppose on the other hand that not every A is B. Then there will be something that is an A but is not B. So it won't be true that everything is such that if it's an A it is B.

Traditionally, the main reservation expressed about this symbolization concerns a possible situation in which there are no A's at all. Suppose that a naturalist is uncertain about whether or not there are any friendly elephants, but is willing to assert:

Every friendly elephant is an herbivore.

Suppose that there are in fact no friendly elephants. Then is what the naturalist said true or false? If we accept the proposed symbolization above, we will represent the naturalist as having said something true. Let us see why this is so. The proposed symbolization is:

$\forall$x(x is a friendly elephant $\rightarrow$ x is an herbivore),

that is:

$\forall x(Fx \wedge Ex \rightarrow Hx)$.

If there are no friendly elephants, this sentence will be true, because, treating `x' as a name letter, the following is true no matter what `x' stands for:

$Fx \wedge Ex \rightarrow Hx$.

It is true because no matter what `x' stands for, the antecedent is false (because there are no friendly elephants).

Is that a proper treatment of the English sentence that was asserted? The consensus on this matter seems to be "sometimes yes, sometimes no." That is, sometimes when we say "*Every A is B*" we presuppose or imply that there are some A's, and sometimes we are neutral on this. In this text we will always take the weaker interpretation, supposing that "*Every A is B*" does not commit you to there being any A's. It is true, not false, if there are no A's. This is just a convention (a widely adopted one) for our convenience. (If you want a version of '*Every A is B*' that does commit you to there being A's, just write: '$\exists xAx \wedge \forall x(Ax \rightarrow Bx)$'.)

The **particular affirmative** form -- "*Some A is B*" -- is easy to symbolize; it gets represented as:

$$\exists x(Ax \wedge Bx),$$

that is, "*Something is such that it is both A and B*."

Plural forms of categorical sentences are symbolized just like the singular forms:

All A's are B	*Every A is B*	$\forall x(Ax \rightarrow Bx)$
Some A's are B	*Some A is B*	$\exists x(Ax \wedge Bx)$

This might seem wrong if you think that the use of the plural in English commits you to the view that there is more than one A which is B. (The symbolized version has no such commitment.) The answer seems to be that we sometimes use the plural to convey the thought that there is more than one A, but sometimes we are neutral about this. In this text we will adopt the weaker interpretation, which makes "*Some A's are B*" true whenever there is at least one A that is B.

The **universal negative** form is:

No A is B

There are two equally natural ways to symbolize this. One way depends on noticing that "*No A is B*" is equivalent to saying "*Every A is not B*," which can be symbolized as:

$$\forall x(Ax \rightarrow \sim Bx).$$

The other way is to notice that "*No A is B*" is equivalent to denying that "*At least one A is B*," and symbolizing the sentence as:

$$\sim\exists x(Ax \wedge Bx).$$

Soon we will be able to prove that these two forms are logically equivalent.

There are two traps to beware of when symbolizing categorical sentences. They both involve trying to make the symbolizations of "universal" and "particular" sentences look alike. Suppose that we want to symbolize:

Some dogs are brown.

It will not be correct to symbolize this as:

$$\exists x(Dx \rightarrow Bx),$$

that is:

Something is such that if it's a dog then it's brown.

This would be wrong because in some possible situations the symbolized version would differ in truth-value from the English version. Consider a possible situation which is just like the actual one except that all dogs are black, white, or grey. The English sentence '*Some dogs are brown*' would be false in that situation. But the symbolized version would be true in that situation. It would be true for the totally irrelevant reason that not everything is a dog!!! Remember the official account of the existential quantifier; the sentence:

$\exists x(Dx \to Bx)$

is true if there is something to let `x' stand for which makes this true:

$Dx \to Bx$.

But that's easy; just let `x' stand for some thing that is not a dog -- and then we have a conditional whose antecedent is false. And such a conditional is true. The symbolized version is automatically true if there is anything that isn't a dog, whereas the English sentence is not automatically true in such a situation. So the symbolization is not a good one to use for that English sentence.

The other trap is to try to symbolize:

Every A is B

as:

$\forall x(Ax \wedge Bx)$.

For example, you might try to symbolize:

Every dog is a mammal

as:

$\forall x(Dx \wedge Mx)$.

It is easy to see that this cannot be a correct symbolization, for the English sentence is true, whereas the symbolized version is false. The symbolized version says:

Everything is such that it is a dog and it is a mammal,

that is:

Everything is both a dog and a mammal.

But *you* are not both a dog and a mammal, so the symbolic sentence is false. So the symbolic sentence is not a correct way to represent the English sentence we are trying to symbolize, `*All dogs are mammals*', since the English sentence is true. The right way to translate the English sentence is the way discussed above:

$\forall x(Dx \to Mx)$.

EXERCISES

1. Symbolize these sentences.

   a. *Every handsome elephant is friendly.*
   b. *No handsome elephant is friendly.*
   c. *Some elephants are not handsome.*
   d. *Some handsome elephants are friendly.*
   e. *Each friendly elephant is handsome.*
   f. *A handsome elephant is not friendly.*
   g. *No friendly elephant is handsome.*

5B COMPLEX CATEGORICAL FORMS

Many sentences are constructed out of categorical forms. An example is:

> *Every brown dog is happy and well-fed*

To symbolize this sentence, notice that the sentence in fact is a universal affirmative sentence; it just happens to have a complex antecedent and a complex consequent. So begin by using the pattern for universal affirmatives:

> ∀x(x *is a brown dog* → x *is happy and well-fed*)

Then complete the symbolization by filling in the details in the antecedent and consequent:

> $\forall x(Bx \wedge Dx \to Hx \wedge Fx)$

(The combination Adjective + Noun, such as '*brown dog*', gets symbolized as a conjunction. For the cases under consideration in this text, that is always the way to symbolize a combination consisting of an adjective modifying a noun.)

This example is similar:

> *Some brown dog isn't either happy or lively.*

Its overall form is that of a particular affirmative:

> ∃x(x *is a brown dog* ∧ x *isn't either happy or lively*)

Its symbolization is then got by filling in the details in the conjuncts:

> $\exists x(Bx \wedge Dx \wedge \sim(Hx \vee Lx))$

Some other examples like this are:

> *No dog is happy unless every dog is well-fed*
> ∀x(x *is a dog* → ~x *is happy*) unless ∀x(x *is a dog* → x *is well-fed*)
> $\forall x(Dx \to \sim Hx) \vee \forall x(Dx \to Fx)$
>
> *Each dog is happy unless it isn't well-fed*
> ∀x(x *is a dog* → x *is happy* unless x *is not well-fed*)
> $\forall x(Dx \to Hx \vee \sim Fx)$

As we have seen, categorical sentences can themselves be combined with connectives. Another example is:

> *If every dog is well-fed, and every dog is an animal, and every animal is happy, then every dog is both well-fed and happy.*

This is a complex of categorical sentences:

> If $\forall x(Dx \to Fx)$ and $\forall y(Dy \to Ay)$ and $\forall z(Az \to Hz)$ then $\forall z(Dz \to Fz \wedge Hz)$

that is:

> $\forall x(Dx \to Fx) \wedge \forall y(Dy \to Ay) \wedge \forall z(Az \to Hz) \to \forall z(Dz \to Fz \wedge Hz)$

Sometimes a sentence is apparently ambiguous, but variable binding resolves the ambiguity. This happens in the example

> *Each dog is happy unless it isn't well-fed*

We decided above to include the '*unless*' as part of the consequent of the quantified conditional. We might try instead to make '*unless*' be the major connective:

$\forall$x(x *is a dog* $\rightarrow$ x *is happy*) unless x *isn't well-fed*
$\forall x(Dx \rightarrow Hx) \lor \sim Fx$

However, this leaves the 'x' unbound by the quantifier. You have a formula that is not a sentence, and there is no way to interpret the unbound occurrence of 'x'. Whenever a symbolization of an ordinary meaningful English sentence ends up with a variable that is not bound by any quantifier, the symbolization will not be correct.

EXERCISES

2. Suppose that `A' stands for `is a U.S. state', `C' for `is a city', `L' for `is a capital', and `E' for `is in the Eastern time zone'. What are the truth values of these sentences?

a. $\forall x(Cx \rightarrow Lx)$
b. $\exists x(Cx \land Lx)$
c. $\exists x(Cx \land Lx \leftrightarrow Ex)$
d. $\forall x(Cx \land Ex \rightarrow Ax)$
e. $\sim\exists x(Ax \land Ex)$
f. $\exists x(Cx \land Ex) \land \exists x(Cx \land \sim Ex)$
g. $\exists x(Cx \land Ex \land Ax)$
h. $\sim\exists x(Cx \land \sim Cx)$

3. Symbolize the following sentences:

a. *All giraffes are spotted.*
b. *All clever giraffes are spotted.*
c. *No clever giraffes are spotted.*
d. *Every giraffe is either spotted or drab.*
e. *Some giraffes are clever.*
f. *Some spotted giraffes are clever.*
g. *Some giraffes are clever and some aren't.*
h. *Some spotted giraffes aren't clever.*
i. *No spotted giraffe is clever but every unspotted one is.*
j. *Every clever spotted giraffe is either wise or foolhardy.*
k. *Either all spotted giraffes are clever, or all clever giraffes are spotted.*
l. *Every clever giraffe is foolhardy.*
m. *If some giraffes are wise then not all giraffes are foolhardy.*
n. *All giraffes are spotted if and only if no giraffes aren't spotted.*
o. *Nothing is both wise and foolhardy.*

5C "ONLY"

In chapter 1 we looked at how 'only' affects the symbolization of conditionals. The same word occurs in connection with quantification. Consider the sentence:

Only dogs are happy

Reflection on what this says indicates that it could be symbolized the same as:

Any non-dog isn't happy

and thus as:

$\forall x(\sim Dx \to \sim Hx)$

But intuitively the sentence is also equivalent to:

Anything that's happy is a dog

$\forall x(Hx \to Dx)$

Fortunately, we will be able to prove later that these two forms are equivalent.

Recall that the effect of '*only*' on '*if*' is to reverse antecedent and consequent. Something like that occurs here too; compare the sentences:

All dogs are happy	$\forall x(Dx \to Hx)$
Only dogs are happy	$\forall x(Hx \to Dx)$

They look pretty much the same except that the antecedent and consequent of the quantified conditional are switched.

Here are some examples of symbolizations of sentences using '*only*':

Dogs can run, but only birds can fly.

$\forall x(Dx \to Cx) \land \forall x(Fx \to Bx)$

Only birds can fly, but not all of them can.

$\forall x(Fx \to Bx) \land \sim\forall x(Bx \to Fx)$

Dogs are happy and frisky; giraffes are happy, but only the well-fed ones are frisky.

$\forall x(Dx \to Hx \land Fx) \land \forall x(Gx \to Hx) \land \forall x(Gx \land Fx \to Ex)$

(Using 'E_' for '_ is well-fed'.)

Notice that the last conjunct is not symbolized as:

$\forall x(Fx \to Gx \land Ex)$

This would say that everything that is frisky is a well-fed giraffe, which is not what is intended. The point is that among giraffes only the well-fed ones are frisky. The last conjunct could also be symbolized as:

$\forall x(Gx \to (Fx \to Ex))$

The word '*only*' can create ambiguity. Consider the sentence:

Only brown dogs are happy

This could be read as saying that everything that is happy is a brown dog:

$\forall x(Hx \to Bx \land Dx)$

or it could be read as saying that among dogs, every happy one is brown:

$\forall x(Dx \rightarrow (Hx \rightarrow Bx))$

Usually we don't notice such ambiguity since it is usually clear from context which is meant. Emphasis also helps; saying "*Only brown dogs are happy*" indicates that among dogs, only the brown ones are happy. Out of context, the sentence is simply ambiguous.

EXERCISES

4. Symbolize these sentences. If a sentence is ambiguous, give all pertinent symbolizations.

   a. Only friendly elephants are handsome
   b. If only elephants are friendly, no giraffes are friendly
   c. Only the brave are fair.
   d. If only elephants are friendly then every elephant is friendly
   e. All and only elephants are friendly.
   f. If every elephant is friendly, only friendly animals are elephants
   g. If any elephants are friendly, all and only giraffes are nasty
   h. Among spotted animals, only giraffes are handsome.
   i. Among spotted animals, all and only giraffes are handsome
   j. Only giraffes frolic if annoyed.

5D RELATIVE CLAUSES

Relative clauses modify nouns, as adjectives do, although relative clauses are typically more complex. There are two sorts of relative clause: restrictive and non-restrictive, illustrated by:

Non-restrictive	*Dogs, which are frisky, are cute*
Restrictive	*Dogs which are frisky are cute*

Non-restrictive relative clauses do not affect the noun they follow; instead they are used to insert a comment in addition to what the main sentence says. The main sentence of the non-restrictive example is that dogs are cute, and the additional comment is that they are frisky. The entire sentence is used to make both of these claims. If we want to capture the whole content of a sentence with a non-restrictive relative clause the best we can do is to conjoin the two claims:

Dogs are frisky $\wedge$ *Dogs are cute* $\quad \forall x(Dx \rightarrow Fx) \wedge \forall x(Dx \rightarrow Cx)$

A restrictive relative clause restricts the content of the noun to which they are adjoined. In the restrictive example above, it is frisky dogs that are said to be cute, not dogs in general. The symbolization is:

Dogs which are frisky are cute $\quad \forall x(Dx \wedge Fx \rightarrow Cx)$

You can usually tell a non-restrictive relative clause, for it is set off from its surroundings by commas before and after it. When there are no commas, the reading is restrictive.

Restrictive relative clauses are like adjectives, in that in logical form they are conjoined with the noun that they modify. In the above example '*dogs which are frisky*' becomes the conjunction '$Dx \wedge Fx$'. When the relative clause is more complex, it gives you something complex to conjoin to the part originating with the noun that is modified. This is seen in:

Every dog which is neither cute nor frisky is not happy.

$\forall x(\underline{Dx \wedge \sim(Cx \vee Fx)} \rightarrow \sim Hx)$

EXERCISES

5. Symbolize these sentences.

a. Every giraffe which frolics is happy
b. Only giraffes which frolic are happy
c. Only giraffes are animals which are long-necked.
d. If only giraffes frolic, every animal which is not a giraffe doesn't frolic.
e. Some giraffe which frolics is long-necked or happy.
f. No giraffe which is not happy frolics and is long-necked.
g. Some giraffe is not both long-necked and happy.

5E IMPLICIT UNIVERSAL QUANTIFIERS

In the symbolizations we have considered so far, symbolic quantifiers have originated naturally from "universal" quantifier words of English. For example, the universal quantifier is often used in symbolizing a sentence with one of the words 'each', 'every', 'all' in it, and the position of the English quantifier word often corresponds to the position of the symbolic quantifier. In 'Every A is B' the English sentence begins with 'every' and its symbolization begins with '$\forall x$'.

Sometimes a universal quantification originates with an English indefinite article 'a' or 'an'. This happens in:

A dog that is well-fed is happy.

This sentence is most naturally treated as conveying a universal claim, that *any* dog that is well-fed is happy:

$\forall x(Dx \wedge Fx \rightarrow Hx)$

This is in spite of the fact that the indefinite article often conveys an existential claim, as in:

A girl left early

$\exists x(Gx \wedge Lx)$

A good test for this is whether the indefinite article can be paraphrased by 'each'; this is natural in the first example, but not in the second.

A more interesting case is when an indefinite article occurs inside a sentence, indicating a universal quantification with scope over the whole sentence. This happens in:

If a dog is well-fed, it is happy

This appears to be a conditional of the form:

a dog is well-fed $\rightarrow$ *it is happy*

But that won't do, since there is nothing to bind the variable that comes from the '*it*' in the consequent. Instead, the indefinite article indicates a universal quantification of *dog*, with the rest of the sentence within its scope. That is, it has the form:

$\forall x$(x *is a dog* $\rightarrow$ (x *is well-fed* $\rightarrow$ x *is happy*))

$\forall x(Dx \rightarrow(Fx \rightarrow Hx))$

This happens in the following two examples as well. In the first:

A giraffe is wise if and only if it's not foolhardy.

This has a logical form something like:

Every giraffe is such that it is wise if and only if it is not foolhardy

$\forall x(Gx \rightarrow (Wx \leftrightarrow \sim Fx))$

This sentence is similar:

A brown dog is frisky only if it is happy

Every brown dog is such that it is frisky only if it is happy

$\forall x(Bx \wedge Dx \rightarrow (Fx \rightarrow Hx))$

The idea that indefinite phrases sometimes correspond to universal quantifiers with wide scope applies also to plural indefinites -- to plural nouns or noun phrases which have no article or quantifier word before them. An example is:

If dogs are well-fed, then they are happy

∀x(x *is a dog* → (x *is well-fed* → x *is happy*))

$\forall x(Dx \rightarrow (Ex \rightarrow Hx))$

EXERCISES

6. Symbolize the following sentences.

a. If a giraffe is happy then it frolics unless it is lame.
b. A monkey frolics unless it is not happy.
c. Among giraffes, only happy ones frolic.
d. All and only giraffes are happy if they are not lame.
e. A giraffe frolics only if it is happy.
f. Only giraffes frolic if happy.
g. All monkeys are happy if some giraffe is.
h. Cute monkeys frolic.
i. Giraffes run and frolic if and only if they are blissful and exultant.
j. If those who are healthy are not lame, then if they are exultant, they will frolic.
k. Only giraffes and monkeys are blissful and exultant.
l. The brave are happy.
m. If a giraffe frolics, then no monkey is blissful unless it is.
n. Giraffes and monkeys frolic if happy.

6 DERIVATIONS WITH QUANTIFIERS

Our first step in including quantificational sentences in derivations is to extend all of the rules from chapters 1 and 2 to include formulas which have free variables. Although we continue to use derivations for arguments consisting entirely of sentences, it will be essential to also allow formulas inside of the derivations.

In this section we introduce three rules for quantifiers.

Rule ui (universal instantiation): The first rule is simple; it says that if everything satisfies a certain condition, any particular thing satisfies that condition. That is, from any universally quantified formula one may infer the result of removing the initial quantifier, and replacing every occurrence of the variable that it was binding by a name letter or by a variable:

Rule ui: (universal instantiation):

$\forall x\ \ldots x \ldots x \ldots$	$\forall x\ \ldots x \ldots x \ldots$
$\therefore\ \ldots b \ldots b \ldots$	$\therefore\ \ldots y \ldots y \ldots$

Every occurrence of 'x' that '∀x' was binding must be replaced with the same name or variable.

An example of this rule is to validate the argument from '*everything is either mental or physical*' to '*Disneyland is either mental or physical*':

$\forall x(Mx \lor Px)$
$\therefore\ Ma \lor Pa$ by rule ui

A more typical application would be to use rule ui to validate an inference like this:

Every giraffe is happy
Fido is a giraffe
∴ *Fido is happy*

$\forall x(Gx \to Hx)$
Gf
$\therefore\ Hf$

A derivation using rule ui to validate this argument could go like this:

1. ~~Show~~ Hf
2. $Gf \to Hf$ pr1 ui
3. Hf pr2 2 mp dd

The universal instantiation step takes us from "everything is such that if it is a giraffe then it is happy" to "if Fido is a giraffe then Fido is happy". Modus ponens does the rest.

In using rule ui the quantifier must be on the front of the formula and it must have scope over the whole formula. If it has a narrower scope, then it is fallacious to apply the rule. For example, this inference is not permitted:

$\forall xFx \to Fg$	*if everything is happy, Gertrude is happy*	(logically true)
$\therefore\ Fb \to Fg$	*if Betty is happy, Gertrude is happy*	(not logically true)

Rule eg (existential generalization): The second rule is the reverse of the first, using the existential quantifier instead of the universal. It says that if a particular thing satisfies a certain condition, then something satisfies it. That is, from any formula one may infer the result of replacing some occurrences of a name letter or a variable in it by a new variable, putting an existential quantifier on the front using that variable.

> **Rule eg (existential generalization):**
>
> | | |
> |---|---|
> | ...b...b... | ...y...y... |
> | ∴ ∃x…x…b… | ∴ ∃x…x…b… |
>
> (You need not replace every occurrence of 'b' or of 'y' by 'x'.)

For example, if Fido is a brown dog, then something is a brown dog:

Bf ∧ Df
∴ ∃x(Bx ∧ Dx)

The existential quantifier that is put on the front must have scope over the whole formula. If the formula you start with is in informal notation, you may need to restore the dropped parentheses before applying the rule, as we did here.

Here is a little derivation that uses both of these rules. It validates the argument:

Every dog is happy
Fido is a dog
∴ *Something is happy*

∀x(Dx → Hx)
Df
∴ ∃xHx

1. ~~Show~~ ∃xHx

2.	Df → Hf	pr1 ui
3.	Hf	2 pr2 mp
4.	∃xHx	3 eg dd

There is a difference between Rules ui and eg. When using rule ui, you must replace every occurrence of the variable that the initial quantifier binds with a name or variable. For example, you cannot do this:

∀x(Dx → Hx)
∴ Dx → Hb

That is:

Everything is such that if it is a dog then it is happy.
∴ *If it is a dog then Bob is happy*

Rule eg is different. When using rule eg you needn't replace all of the occurrences. For example, from:

Bob is happy or Bob is sad

you may infer

Something is such that Bob is happy or it is sad.

This conclusion looks odd, but it should be clear that it follows logically.

Here is another example of a derivation using both of our new rules:

Fido is a dog
Every dog is happy
∴ *Some dog is happy*

1. ~~Show~~ $\exists x(Dx \wedge Hx)$

2.	$Df \rightarrow Hf$	pr2 ui
3.	Hf	2 pr1 mp
4.	$Df \wedge Hf$	pr1 3 adj
5.	$\exists x(Dx \wedge Hx)$	4 eg dd

There is a constraint on both of these rules: there must be no "capturing". If a new variable appears in the conclusion of either rule that was not there previously, it must not be "captured" by a quantifier in the formula. Specifically, if a new variable appears, none of its new occurrences may be bound by a quantifier already in the formula. For example, this use of rule eg is not permitted:

$Df \wedge \forall x(Hf \rightarrow Gx)$
∴ $\exists x(Dx \wedge \forall x(Hx \rightarrow Gx))$ ← the universal quantifier captures the variable 'x' that replaces the second 'f'

> **No capturing:**
>
> When using rule ui or rule eg a new variable must not be introduced if some of its new occurrences are bound by a quantifier in the original formula.

You will not often encounter cases of capturing; they usually happen by accident. The possibility of capturing can be avoided by always choosing a variable that does not already occur in the formula.

EXERCISES

1. Symbolize these arguments and produce derivations for them.

a. *The sky is blue*
Everything that is blue is pretty
∴ *Something is pretty*

b. Every hyena is grey.
Every hyena is an animal
Jenny is a hyena
∴ Some animal is grey

c. If some hyena is grey, every hyena is grey
Every scavenger is grey
Jenny is a hyena and a scavenger
Kathy is a hyena
∴ Kathy is grey

Rule ei (existential instantiation): Our third rule is rule ei (existential instantiation). It works just like universal instantiation, except that (1) it applies to an existential quantifier, (2) you must instantiate to a variable, not to a name letter, and (3) you must use a variable that has not already occurred in the derivation or in any of the premises that have been cited in the derivation. This rule is meant to capture the following kind of reasoning. Suppose that you are given the information:

Every dog is happy
Something is a dog

and you wish to infer that something is happy. You are not told that any particular named thing is a dog; you just know that there are some. You might reason as follows:

> By the second premise, there are some dogs. Call one of them "z". Then z is happy (by the first premise), so something is happy.

What you did was to choose a label, 'z', for some dog, without specifying which dog it is. Then you made inferences using that label, ending up with a conclusion that does not contain the label. The label was just a device to reason with.

It was important that you chose a label that was not already assigned to something. If you used an already existing name for the label, that could lead to fallacies. For example, consider this bad argument:

Every dog is happy
Something is a dog
Fluffy is a cat
∴ *Some cat is happy*

It would be wrong to reason like this:

> By the second premise, there are some dogs. Call one of them "Fluffy". Then Fluffy is happy (by the first premise). Also, Fluffy is a cat (third premise). So some cat is happy.

By using the name 'Fluffy' for one of the dogs you were implicitly assuming that Fluffy was a dog. That assumption is not justified. Formally we get around such an unjustified assumption by using only variables for labels, and by requiring that these variables are not already used for something else. We accomplish this by requiring that the new variable not have occurred already in the derivation:

Rule ei: (existential instantiation):

$\exists x$ ...x...x...
∴ …y…y…

You must replace every occurrence of 'x' that '$\exists x$' was binding.
The variable 'y' must not occur in the existentially quantified formula itself, or in previous lines in the derivation, or in a premise that has been cited on a previous line.

Here now is a derivation using all of our new rules:

	$\forall x(Bx \wedge Dx \rightarrow Ex)$	*Every brown dog is well-fed.*
	$\exists x(Dx \wedge Fx)$	*Some dog is frisky*
	$\forall y(Fy \rightarrow By)$	*Everything frisky is brown*
∴	$\exists z(Dz \wedge Ez)$	∴ *Some dog is well-fed*

1.	~~Show~~ ∃z(Dz ∧ Ez)		
2.	Du ∧ Fu	pr2 ei	('u' has not already occurred in the derivation)
3.	Fu → Bu	pr3 ui	
4.	Bu	2 s 3 mp	
5.	Bu ∧ Du	2 s 4 adj	
6.	Bu ∧ Gu → Eu	pr1 ui	
7.	Eu	5 6 mp	
8.	Du ∧ Eu	2 s 7 adj	
9.	∃z(Dz ∧ Ez)	8 eg dd	

The reader should check to see that each of the new rules is properly used.

This derivation illustrates an important strategy rule. Often you will have an opportunity to apply ei to introduce a variable, and then use ui to instantiate to that variable. In the derivation just given, ei introduces 'u' on line 2 and ui is used twice to instantiate to 'u', on lines 3 and 6. The strategy rule is that when this is a possibility, you should always apply rule ei before you apply rule ui.

> **Strategy hint**: When using both ei and ui to instantiate to the same variable, apply rule ei before rule ui.

This is because if you try using ui first, you will not then be able to use ei to instantiate to the same variable, because the variable will not then be new. For example, suppose that you started the above derivation with:

1.	Show ∃z(Dz ∧ Ez)	
2.	Fu → Bu	pr3 ui
3.	Du ∧ Fu	pr2 ei

Line 3 is fallacious because you have instantiated to 'u', but 'u' has already occurred in the derivation, which violates the constraint that the variable used in ei must be new.

Here is a straightforward illustration of our three rules:

Every crook who steals a lot and doesn't get caught is affluent..
No crook who gets caught is affluent.
Some lucky crooks steal a lot.
Some crooks who aren't lucky don't steal a lot.
Every crook who isn't lucky gets caught.
Every crook who is lucky doesn't get caught.
∴ *Some crooks are affluent and some aren't.*

∀x(Cx ∧ Ex ∧ ~Gx → Ax)
∀x(Cx ∧ Gx → ~Ax)
∃x(Lx ∧ Cx ∧ Ex)
∃x(Cx ∧ ~Lx ∧ ~Ex)
∀x(Cx ∧ ~Lx → Gx)
∀x(Cx ∧ Lx → ~Gx)
∴ ∃x(Cx ∧ Ax) ∧∃x(Cx ∧ ~Ax)

(In doing this derivation recall that 'P ∧ Q ∧ R' is informal notation for '((P ∧ Q) ∧R)'.)

1.	~~Show~~ $\exists x(Cx \wedge Ax) \wedge \exists x(Cx \wedge {\sim}Ax)$	
2.	$Lz \wedge Cz \wedge Sz$	pr3 ei
3.	$Cz \wedge Lz \rightarrow {\sim}Gz$	pr6 ui
4.	Sz	2 s
5.	Lz	2 s s
6.	Cz	2 s s
7.	${\sim}Gz$	5 6 adj 3 mp
8.	$Cz \wedge Sz \wedge {\sim}Gz \rightarrow Az$	pr1 ui
9.	Az	6 4 adj 7 adj 8 mp
10.	$Cz \wedge Az$	6 9 adj
11.	$\exists x(Cx \wedge Ax)$	10 eg
12.	$Cu \wedge {\sim}Lu \wedge {\sim}Eu$	pr4 ei
13.	$Cu \wedge {\sim}Lu$	12 s
14.	$Cu \wedge {\sim}Lu \rightarrow Gu$	pr5 ui
15.	Gu	13 14 mp
16.	$Cu \wedge Gu \rightarrow {\sim}Au$	pr2 ui
17.	${\sim}Au$	13 s 15 adj 16 mp
18.	$Cu \wedge {\sim}Au$	13 s 17 adj
19.	$\exists x(Cx \wedge {\sim}Ax)$	18 eg
20.	$\exists x(Cx \wedge Ax) \wedge \exists x(Cx \wedge {\sim}Ax)$	11 19 adj dd

Notice that the ei step in line 2 precedes the ui steps in lines 3 and 8, and that the ei step in line 12 precedes the ui steps in lines 14 and 16.

EXERCISES

2. Here is a fallacious derivation to validate this argument:

	$\exists x(Nx \wedge Ex)$	*some number is even*
	$\exists x(Nx \wedge Ox)$	*some number is odd*
$\therefore$	$\exists x(Nx \wedge Ox \wedge Ex)$	*some number is both odd and even*

Identify the error in the derivation.

1.	~~Show~~ $\exists x(Nx \wedge Ox \wedge Ex)$	
2.	$Nz \wedge Ez$	pr1 ei
3.	$Nz \wedge Oz$	pr2 ei
4.	$Nz \wedge Oz \wedge Ez$	2 s 3 adj
5.	$\exists x(Nx \wedge Ox \wedge Ex)$	dd

3. Produce derivations for each of the following (be careful to obey the strategy rule just given):

a. theorem T202: $\therefore\ \forall x(Fx \rightarrow Gx) \rightarrow (\exists xFx \rightarrow \exists xGx)$
b. half of T203: $\therefore\ \exists x{\sim}Fx \rightarrow {\sim}\forall xFx$
c. half of T204: $\therefore\ \forall x{\sim}Fx \rightarrow {\sim}\exists xFx$

T201 $\forall x(Fx \rightarrow Gx) \rightarrow (\forall xFx \rightarrow \forall xGx)$
T202 $\forall x(Fx \rightarrow Gx) \rightarrow (\exists xFx \rightarrow \exists xGx)$
T203 ${\sim}\forall xFx \leftrightarrow \exists x{\sim}Fx$
T204 ${\sim}\exists xFx \leftrightarrow \forall x{\sim}Fx$

7 UNIVERSAL DERIVATIONS

We have two instantiation rules, one for each quantifier, and we have a generalization rule for the existential quantifier. It is customary and useful to have some kind of universal generalization rule as well. For example, someone might want to reason as follows:

Every dog is a mammal
Every mammal is an animal
∴ *Every dog is an animal*

A natural approach might be like this. Let z be anything whatsoever. Instantiating the first premise tells us that if z is a dog, it is a mammal; and instantiating the second premise tells us that if z is a mammal, it is an animal. So using techniques from chapter 1, we may infer that if z is a dog, z is an animal. Now since 'z' was chosen to represent anything whatever, we can infer that everything is such that if it is a dog it is an animal. That is, every dog is an animal.

What we want to capture is the idea that if you can show something for any arbitrarily chosen thing, it holds for everything. Something like:

$Dz \to Az$
∴ $\forall x(Dx \to Ax)$ because z is anything at all

For technical reasons, this principle will be formulated not as a rule, but as a special kind of derivation. It will take the form that if you want to show a universal claim, and you succeed in showing that it holds for a variable, z, then if z is completely arbitrary, you may box and cancel the show line for the universal claim. So the above reasoning will take this form: If you have a derivation of this form:

Show $\forall x(Dx \to Ax)$
:::::
:::::
$Dz \to Az$ where z is completely arbitrary

Then you can box and cancel

~~Show~~ $\forall x(Dx \to Ax)$
:::::
:::::
$Dz \to Az$ ud where z is completely arbitrary

The requirement that z be completely arbitrary is realized by the technical requirement that 'z' shall not have occurred free anywhere in the derivation above the show line.

The "ud" notation is the name of our new form of derivation:

Universal derivation:

If you have a derivation of the following form:

Show $\forall$x . . . x . . . x . . .
:::::
:::::
. . . x . . . x . . .

Then if there are no uncancelled show lines in between the first and last lines displayed, and if 'x' does not occur free on any line in the derivation available from the show line, or in any premise cited in an available line, you may box and cancel, using the notation 'ud'.

The reasoning suggested above may now be incorporated into a derivation like this:

	$\forall$x(Dx $\rightarrow$ Mx)		*Every dog is a mammal*
	$\forall$y(My $\rightarrow$ Ay)		*Every mammal is an animal*
∴	$\forall$z(Dz $\rightarrow$ Az)	∴	*Every dog is an animal*

1. ~~Show~~ $\forall$z(Dz $\rightarrow$ Az)
2. Dz $\rightarrow$ Mz   pr1 ui
3. Mz $\rightarrow$ Az   pr2 ui
4. ~~Show~~ Dz $\rightarrow$ Az
5.   Dz   ass cd
6.   Mz   2 5 mp
7.   Az   3 6 mp   cd
8.   4 ud

The reader should check that this derivation meets the conditions necessary for a ud derivation.

In a previous exercise we proved half of theorem 203. The other half of T203 is more difficult.

∴ ~$\forall$xFx $\rightarrow$ $\exists$x~Fx

It is easy to begin the derivation:

1. Show ~$\forall$xFx $\rightarrow$ $\exists$x~Fx
2.   ~$\forall$xFx   ass cd
3.   ?????

With no other guide, our strategy rules say to try id:

1. Show ~$\forall$xFx $\rightarrow$ $\exists$x~Fx
2.   ~$\forall$xFx   ass cd
3.   Show $\exists$x~Fx
4.   ~$\exists$x~Fx   ass id
5. ???

Again there is no clear way to proceed. Since we are trying to derive any contradiction, we try to derive the unnegation of line 2:

1. Show $\sim\forall xFx \rightarrow \exists x\sim Fx$
2. $\sim\forall xFx$ ass cd
3. Show $\exists x\sim Fx$
4. $\sim\exists x\sim Fx$ ass id
5. Show $\forall xFx$
6. ?????

Since we are trying to show a universally quantified formula, it is natural to try to show step 5 by means of a universal derivation. We only need to show Fx.

1. Show $\sim\forall xFx \rightarrow \exists x\sim Fx$
2. $\sim\forall xFx$ ass cd
3. Show $\exists x\sim Fx$
4. $\sim\exists x\sim Fx$ ass id
5. Show $\forall xFx$
6. Show Fx
7. ???

It is easy now to show Fx by means of another indirect derivation:

1. Show $\sim\forall xFx \rightarrow \exists x\sim Fx$
2. $\sim\forall xFx$ ass cd
3. Show $\exists x\sim Fx$
4. $\sim\exists x\sim Fx$ ass id
5. Show $\forall xFx$
6. Show Fx
7. $\sim Fx$ ass id
8. $\exists x\sim Fx$ 7 eg
9. $\sim\exists x\sim Fx$ 4 r

This completes the indirect derivation, so we box and cancel. We have now completed each of the other subderivations, so we box and cancel them too. The result is:

1. ~~Show~~ $\sim\forall xFx \rightarrow \exists x\sim Fx$
2. $\sim\forall xFx$ ass cd
3. ~~Show~~ $\exists x\sim Fx$
4. $\sim\exists x\sim Fx$ ass id
5. ~~Show~~ $\forall xFx$
6. ~~Show~~ Fx
7. $\sim Fx$ ass id
8. $\exists x\sim Fx$ 7 eg
9. $\sim\exists x\sim Fx$ 4 r 8 id
10. 6 ud
11. $\sim\forall xFx$ 2 r 6 id
12. 3 cd

EXERCISES

1. Produce derivations for each of the following (be careful to obey the strategy rule just given):

   a. theorem T201: $\therefore\ \forall x(Fx \rightarrow Gx) \rightarrow (\forall xFx \rightarrow \forall xGx)$
   b. half of T204: $\therefore\ \sim\exists xFx \rightarrow \forall x\sim Fx$ (similar to the derivation of half of T203)
   c. half of theorem T205: $\therefore\ \forall zFx \rightarrow \sim\exists x\sim Fx$

8 SOME DERIVATIONS

Many derivations take a common form. You begin with quantified sentences, and you remove quantifiers. Then you manipulate formulas using the techniques from chapters 1 and 2. Finally, you restore the quantifiers. In some cases this is straightforward:

Every bear is friendly
Some bear is dangerous
∴ *Something dangerous is friendly*

$\forall x(Px \rightarrow Qx)$
$\exists y(Py \wedge Ry)$
$\therefore\ \exists z(Rz \wedge Qz)$

First we remove quantifiers using instantiation rules, being careful to apply ei before ui when that is possible:

1.	Show $\exists z(Rz \wedge Qz)$	
2.	$Pu \wedge Ru$	pr2 ei
3.	$Pu \rightarrow Qu$	pr1 ui

We choose to use 'y' in the universal instantiation step because it gives us something useful. Choosing other variables or names would be correct, but not useful.

Now we use sentential rules to get a formula that we can existentially quantify:

4.	Qu	2 s 3 mp
5.	$Ru \wedge Qu$	2 s 4 adj

Now we are in a position to existentially quantify line 5 to get the desired conclusion:

6.	$\exists z(Rz \wedge Qz)$	5 eg

We can then box and cancel:

1.	~~Show~~ $\exists z(Rz \wedge Qz)$	
2.	$Pu \wedge Ru$	pr2 ei
3.	$Pu \rightarrow Qu$	pr1 ui
4.	Qu	2 s 3 mp
5.	$Ru \wedge Qu$	2 s 4 adj
6.	$\exists z(Rz \wedge Qz)$	5 eg dd

Strategy hint: When a line is available that begins with a universal or existential quantifier, apply an instantiation rule, ei or ui, to derive an instance.

When the conclusion is a universally quantified formula, it will very likely be derived by using a universal derivation. When a universal derivation is used, it is usually best to set up the derivation as early as possible. Consider this example:

Every jaguar is a fast cat
Every cat is an animal
∴ *Every jaguar is a fast animal.*

$\forall x(Jx \rightarrow Fx \wedge Cx)$
$\forall x(Cx \rightarrow Ax)$
∴ $\forall x(Jx \rightarrow Fx \wedge Ax)$

Our initial show line is a universally quantified sentence:

1. Show $\forall x(Jx \rightarrow Fx \wedge Ax)$

We can derive line 1 if we can show the formula that you get by removing its initial quantifier. So set that up as a show line:

2. Show $Jx \rightarrow Fx \wedge Ax$

This is a conditional, so try conditional derivation:

3.	Jx	ass cd

The rest of the conditional derivation is relatively straightforward:

4.	$Jx \rightarrow Fx \wedge Cx$	pr1 ui
5.	$Fx \wedge Cx$	3 4 mp
6.	$Cx \rightarrow Ax$	pr2 ui
7.	Ax	5 s 6 mp
8.	$Fx \wedge Ax$	5 s 7 adj

We have derived the consequent of the conditional to be shown; after boxing and canceling we have:

1.	Show $\forall x(Jx \rightarrow Fx \wedge Ax)$	
2.	~~Show~~ $Jx \rightarrow Fx \wedge Ax$	
3.	Jx	ass cd
4.	$Jx \rightarrow Fx \wedge Cx$	pr1 ui
5.	$Fx \wedge Cx$	3 4 mp
6.	$Cx \rightarrow Ax$	pr2 ui
7.	Ax	5 s 6 mp
8.	$Fx \wedge Ax$	5 s 7 adj cd

Since line 2 has been shown, we may infer line 1 by universal derivation:

1.	~~Show~~ $\forall x(Jx \rightarrow Fx \wedge Ax)$	
2.	~~Show~~ $Jx \rightarrow Fx \wedge Ax$	
3.	Jx	ass cd
4.	$Jx \rightarrow Fx \wedge Cx$	pr1 ui
5.	$Fx \wedge Cx$	3 4 mp
6.	$Cx \rightarrow Ax$	pr2 ui
7.	Ax	5 s 6 mp
8.	$Fx \wedge Ax$	5 s 7 adj cd
9.		2 ud

When the conclusion has both universal and existential quantifiers, the strategy is essentially to combine those above, applying whichever strategy is relevant at the time. Consider this argument:

For every giraffe, there is a leopard which is happy if and only if it (the giraffe) is.
For every leopard, there is a monkey that is happy if and only if it (the leopard) is.
∴ For every giraffe, there is a monkey which is happy if and only if it (the giraffe) is.

∀x(Gx → ∃y(Ly ∧ (Hy ↔ Hx)))
∀x(Lx → ∃y(My ∧ (Hy ↔ Hx)))
∴ ∀x(Gx → ∃y(My ∧ (Hy ↔ Hx)))

The conclusion to be shown is universally quantified, so set up a universal derivation. In fact, this should generally be done as early as possible.

> **Strategy hint**: If a universal derivation is to be used to show a universally quantified formula, ∀x□, set it up as early as possible, by inserting a Show line containing the formula, □, following the quantifier.

This is done in line 2 here:

1. Show ∀x(Gx → ∃y(My ∧ (Hy ↔ Hx)))
2.   Show Gx → ∃y(My ∧ (Hy ↔ Hx))

Line 2 is a conditional, so try conditional derivation:

3.	Gx	ass cd

Universally instantiating the first premise and using modus ponens is a natural thing to try:

4.	Gx → ∃y(Ly ∧ (Hy ↔ Hx))	pr1 ui
5.	∃y(Ly ∧ (Hy ↔ Hx))	3 4 mp

We now have derived an existentially quantified formula, and there are some universally quantified ones in the premises. Generally, when both rules ei and ui are possible, as we stated above, you should use rule ei first. This is because rule ei introduces a variable which must be brand new in the derivation. If you do ei first, then you can do ui using the variable introduced by ei. But if you do ui first, you cannot do ei using that variable. In our derivation, the "ei before ui" strategy is relevant. Apply ei to line 5 using a variable that does not already occur in the derivation:

6.	Lz ∧ (Hz ↔ Hx)	

We can now make use of our second premise to get:

7.	Lz → ∃y(My ∧ (Hy ↔ Hz))	pr2 ui

We can obviously use line 6 to get the consequent of line 7. That consequent is also existentially quantified, so we apply ei:

8.	∃y(My ∧ (Hy ↔ Hz))	6 s 7 mp
9.	Mu ∧ (Hu ↔ Hz)	8 ei

Now look over what we have and what we want. We are in a conditional derivation, and we need to show '∃y(My ∧ (Hy ↔ Hx))' to complete that derivation. This formula is existentially quantified, and so we will probably derive it by existentially generalizing something. That is, we will existentially generalize something of the form:

M_ ∧ (H_ ↔ Hx)

We already have something very close to that, on line 9; we could get what we want by deriving a formula just like line 9 but with 'x" instead of 'z'. So suppose we try to derive ' $Mu \wedge (Hu \leftrightarrow Hx)$'. We already have the left conjunct, so the job is to derive the right conjunct '$Hu \leftrightarrow Hx$'. This is a biconditional, so we need to derive two conditionals, probably by conditional derivation, and then put them together by cb. That in fact is easy to do:

10.	~~Show~~ $Hu \rightarrow Hx$	
11.	Hu	ass cd
12.	Hz	9 s bc 11 mp
13.	Hx	6 s bc 12 mp cd
14.	~~Show~~ $Hx \rightarrow Hu$	
15.	Hx	ass cd
16.	Hz	6 s bc 15 mp
17.	Hu	9 s bc 16 mp cd
18.	$Hu \leftrightarrow Hx$	10 14 cb

To finish, we only need to put line 18 together with the first conjunct on line 9, and existentially generalize:

19.	$Mu \wedge (Hu \leftrightarrow Hx)$	9 s 18 adj
20.	$\exists y(My \wedge (Hy \leftrightarrow Hx))$	19 eg

This completes our conditional derivation, so we now have:

1.	Show $\forall x(Gx \rightarrow \exists y(My \wedge (Hy \leftrightarrow Hx)))$	
2.	~~Show~~ $Gx \rightarrow \exists y(My \wedge (Hy \leftrightarrow Hx))$	
3.	Gx	ass cd
4.	$Gx \rightarrow \exists y(Ly \wedge (Hy \leftrightarrow Hx))$	pr1 ui
5.	$\exists y(Ly \wedge (Hy \leftrightarrow Hx))$	3 4 mp
6.	$Lz \wedge (Hz \leftrightarrow Hx)$	5 ei
7.	$Lz \rightarrow \exists y(My \wedge (Hy \leftrightarrow Hz))$	pr2 ui
8.	$\exists y(My \wedge (Hy \leftrightarrow Hz)$	6 s 7 mp
9.	$Mu \wedge (Hu \leftrightarrow Hz)$	8 ei
10.	~~Show~~ $Hu \rightarrow Hx$	
11.	Hu	ass cd
12.	Hz	9 s bc 11 mp
13.	Hx	6 s bc 12 mp cd
14.	~~Show~~ $Hx \rightarrow Hu$	
15.	Hx	ass cd
16.	Hz	6 s bc 15 mp
17.	Hu	9 s bc 16 mp cd
18.	$Hu \leftrightarrow Hx$	10 14 cb
19.	$Mu \wedge (Hu \leftrightarrow Hx)$	9 s 18 adj
20.	$\exists y(My \wedge (Hy \leftrightarrow Hx))$	19 eg cd

Line 2 has now been shown by the conditional derivation. Now we only need to add line 21, and box and cancel, finishing the universal derivation.

1. ~~Show~~ ∀x(Gx → ∃y(My ∧ (Hy ↔ Hx)))
2. ~~Show~~ Gx → ∃y(My ∧ (Hy ↔ Hx))

   [[DETAILS ABOVE]]

21. 2 ud

EXERCISES

1. Symbolize these arguments and provide derivations to validate them. Give an explicit scheme of abbreviation for each.

a. *If history is right, then if anyone was strong, Hercules was strong.*
Only those who work out are strong, and only those with self-discipline work out.
∴ *If Hercules does not have self-discipline, then either history is not right or nobody is strong.*

b. *If some giraffes are not happy, then all giraffes are morose.*
Some giraffes ponder the mysteries of life.
∴ *If some giraffes are not morose, then some who ponder the mysteries of life are happy.*

c. *There is not a single critic who either likes art or can paint.*
Some level-headed people are critics.
Anyone who can't paint is uneducated.
∴ *Some level-headed people are uneducated.*

d. *No astronaut is a good dancer.*
Every singer is warm-blooded.
If something is warm-blooded and is not a good dancer, then nothing that is either a singer or an astronaut is exultant.
∴ *If some astronaut is a singer, then no singer is exultant.*

e. *All students who have a sense of humor or are brilliant seek fame.*
Anyone who seeks fame and is brilliant is insecure.
Whoever is a mogul is brilliant.
∴ *Every student who is a mogul.*

f. *There is a monkey that is happy if and only if some giraffe is happy.*
There is a monkey that is happy if and only if some giraffe is not happy.
All monkeys are happy.
∴ *It is not the case that either every giraffe is happy or none are.*

g. *For every astronaut that writes poetry, there is one that doesn't.*
For every astronaut that doesn't write poetry, there is one that does.
∴ *If there are any astronauts, some write poetry and some don't.*

2. Derive theorems 203-208, 231-232.

T203 **~∀xFx ↔ ∃x~Fx**
T204 **~∃xFx ↔ ∀x~Fx**
T205 **∀xFx ↔ ~∃x~Fx**
T206 **∃xFx ↔ ~∀x~Fx**

T207 $\exists x(Fx \lor Gx) \leftrightarrow \exists xFx \lor \exists xGx$
T208 $\forall x(Fx \land Gx) \leftrightarrow \forall xFx \land \forall xGx$
T209 $\exists x(Fx \land Gx) \rightarrow \exists xFx \land \exists xGx$
T210 $\forall xFx \lor \forall xGx \rightarrow \forall x(Fx \lor Gx)$

::::::::

T231 $\forall xFx \leftrightarrow \forall yFy$
T232 $\exists xFx \leftrightarrow \exists yFy$

9 DERIVED RULES

We have looked at formulas that have quantifiers on their front, or quantifiers that end up on front after a step such as modus ponens. Things are different if those quantifiers are preceded by a negation sign. Consider the following simple derivation:

Every A is B.
Nothing is both B and C.
So every A isn't C.

$\forall x(Ax \rightarrow Bx)$
$\sim\exists x(Bx \land Cx)$
$\therefore$ $\forall x(Ax \rightarrow \sim Cx)$

This is intuitively valid, but deriving it requires slightly indirect reasoning. Our conclusion is universally quantified, so we set up a universal derivation right away:

1. Show $\forall x(Ax \rightarrow \sim Cx)$
2. Show $Ax \rightarrow \sim Cx$

This is a conditional, so we try conditional derivation:

3.	Ax	ass cd

We can spell out some obvious consequences of what we have by instantiating the first premise and doing modus ponens:

4.	$Ax \rightarrow Bx$	pr1 ui
5.	Bx	3 4 mp

The second premise in fact is not anything we can make use of by applying any of our quantifier rules. Some other approach is needed. At this point it is useful to fall back on a technique from chapter 1; we are trying to derive '~Cx", so try to derive it by indirect derivation:

6.	Show ~Cx	
7.	Cx	ass id

We are not in a position to use the second premise directly, but we can use it indirectly by deriving something that contradicts it. This is simple in two lines:

8.	$Bx \land Cx$	5 7 adj
9.	$\exists x(Bx \land Cx)$	8 eg

Now we complete our indirect derivation with:

10.	$\sim\exists x(Bx \land Cx)$	pr2 9 id

boxing and canceling to get:

1.	Show $\forall x(Ax \to \sim Cx)$	
2.	Show $Ax \to \sim Cx$	
3.	Ax	ass cd
4.	$Ax \to Bx$	pr1 ui
5.	Bx	3 4 mp
6.	~~Show~~ $\sim Cx$	
7.	Cx	ass id
8.	$Bx \wedge Cx$	5 7 adj
9.	$\exists x(Bx \wedge Cx)$	8 eg
10.	$\sim\exists x(Bx \wedge Cx)$	pr2 9 id

This essentially completes the derivation. For line 6 has completed the conditional derivation that starts on line 2, and once the 'show' on line 2 is cancelled, line 1 follows by universal derivation:

1.	~~Show~~ $\forall x(Ax \to \sim Cx)$	
2.	~~Show~~ $Ax \to \sim Cx$	
3.	Ax	ass cd
4.	$Ax \to Bx$	pr1 ui
5.	Bx	3 4 mp
6.	~~Show~~ $\sim Cx$	
7.	Cx	ass id
8.	$Bx \wedge Cx$	5 7 adj
9.	$\exists x(Bx \wedge Cx)$	8 eg
10.	$\sim\exists x(Bx \wedge Cx)$	pr2 9 id
11.		6 cd
12.		2 ud

This kind of indirect strategy is typical of how to handle derivations with sentences that begin with negated quantifiers when we use only our basic rules for quantifiers. However, it is usually more useful to use some derived rules that let us replace initial negated quantifiers by unnegated ones of the opposite sort, which may be used directly. The rule called **quantifier negation** does this. It lets you replace a negated initial quantifier by the opposite quantifier followed by a negation. If we lump in all applications of double negation, we get eight cases:

Rule qn (Quantifier negation)

$\sim\forall xFx$	$\sim\exists xFx$	$\forall xFx$	$\exists xFx$
$\therefore\ \exists x\sim Fx$	$\therefore\ \forall x\sim Fx$	$\therefore\ \sim\exists x\sim Fx$	$\therefore\ \sim\forall x\sim Fx$
$\sim\forall x\sim Fx$	$\sim\exists x\sim Fx$	$\forall x\sim Fx$	$\exists x\sim Fx$
$\therefore\ \exists xFx$	$\therefore\ \forall xFx$	$\therefore\ \sim\exists xFx$	$\therefore\ \sim\forall xFx$

These derived rules are based on T203-206, which are given in the last set of exercises.

Here is how we can use rule qn to shorten the derivation above. We begin as before:

$\forall x(Ax \to Bx)$
$\sim\exists x(Bx \wedge Cx)$
$\therefore\ \forall x(Ax \to \sim Cx)$

1.	Show $\forall x(Ax \to {\sim}Cx)$	
2.	Show $Ax \to Cx$	
3.	Ax	ass cd
4.	$Ax \to Bx$	pr1 ui
5.	Bx	3 4 mp

Now instead of introducing a subderivation to make indirect use of the second premise, we apply rule qn to that premise and then make direct use of the result; this lets us proceed quickly to get the desired '~Cx':

1.	~~Show~~ $\forall x(Ax \to {\sim}Cx)$		
2.	~~Show~~ $Ax \to {\sim}Cx$		
3.	Ax	ass cd	
4.	$Ax \to Bx$	pr1 ui	
5.	Bx	3 4 mp	
6.	$\forall x{\sim}(Bx \wedge Cx)$	pr2 qn	←
7.	${\sim}(Bx \wedge Cx)$	8 ui	
8.	${\sim}Bx \vee {\sim}Cx$	9 dm	
9.	${\sim}Cx$	5 dn 8 mtp cd	
10.		2 ud	

The advantage is not just that the derivation is two lines shorter, but the reasoning is simpler, and it is easier to think up. For that reason we have this strategy hint:

Strategy hint: If an available formula begins with a negation sign immediately followed by a quantifier which has scope over the rest of the formula, convert it to a more useful formula by applying rule qn to it.

Here is another example of the use of rule qn. We are given this argument to validate:

$${\sim}\exists x(Ax \wedge Bx)$$
$$\forall y(Ay \leftrightarrow {\sim}Cy)$$
$$\forall y(Dy \to By)$$
$${\sim}\forall xCx$$
$$\therefore\ \exists x{\sim}Dx$$

Neither the first nor the fourth premise may be used as an input to one of the basic quantifier rules. However, rule qn turns them into useful forms.

1.	~~Show~~ $\exists x{\sim}Dx$		
2.	$\exists x{\sim}Cx$	pr4 qn	←
3.	${\sim}Ck$	2 ei	
4.	$Ak \leftrightarrow {\sim}Ck$	pr2 ui	
5.	Ak	4 bc 3 mp	
6.	$\forall x{\sim}(Ax \wedge Bx)$	pr1 qn	←
7.	${\sim}(Ak \wedge Bk)$	6 ui	
8.	${\sim}Ak \vee {\sim}Bk$	7 dm	
9.	${\sim}Bk$	5 dn 8 mtp	
10.	$Dk \to Bk$	pr3 ui	
11.	${\sim}Dk$	9 10 mt	
12.	$\exists x{\sim}Dx$	11 eg dd	

Rule av: There is another useful derived rule, though one not so often used. Given our explanation of quantifiers, our choice of bound variables is irrelevant; one is as good as another. This is made explicit in derived rule **av** ("alphabetic variance"). The rule says that alphabetically varying the choice of a bound variable used with an initial quantifier yields an equivalent formula. In particular:

> **Rule av** (alphabetic variance)
>
> From a formula of the form '∀x . . . x . . x . . .', where the initial quantifier has scope over the whole formula, you may infer '∀y . . . y . . y . . .', which is the result of changing the variable 'x' in the quantifier to another variable, 'y', and changing all variables inside the first formula that are bound by the initial quantifier to 'y'.
>
> Likewise if the initial quantifier is '∃' instead of '∀'.
>
> Constraint: No capturing is allowed. That is, this inference is not permitted if the new variable becomes bound by a quantifier inside of the original formula.

As an example, from

∀z(Dz ∧ Ez → ∃u(Du ∨ Fz))

you may infer

∀w(Dw ∧ Ew → ∃u(Du ∨ Fw)).

But you may not infer

∀u(Du ∧ Eu → ∃u(Du ∨ Fu))

because that violates the no capturing rule.

Rule av is based on theorems T231 and T232, proved in the exercises.

Here is a situation in which rule av is useful. Suppose you are given the argument:

∀z(Dz ∧ Ez → ∃u(Du ∨ Fz))
∀x(Dx → ~Fx)
∴ ∀u(Du → ~Eu)

A natural derivation might go like this. The conclusion is universally quantified, so set up a universal derivation:

1. Show ∀u(Du → ~Eu)
2. Show Du → ~Eu

This is a conditional, so set up a conditional derivation:

3.	Du	ass cd

You now need to show ~Eu, and it is natural to set up an indirect derivation to show this:

4.	Show ~Eu	
5.	Eu	ass id

Now universally instantiate the first premise:

6.	Du ∧ Eu → ∃u(Du ∧ Fu)	pr1 ui

Oops, you can't do that! The 'u' following the 'F' gets captured by the quantifier in the consequent of the conditional. So what can we do? Different ideas might be tried, but here is an easy one, using rule av. Don't start out to derive the conclusion, because it uses a variable that gets you in trouble. Instead, derive a sentence that is exactly like the conclusion, but one that uses a different variable. Then use rule av to change this into the desired conclusion.

Here is a derivation which reaches a sentence just like the conclusion except for using a different variable:

1. Show $\forall u(Du \to \sim Eu)$
2. ~~Show~~ $\forall w(Dw \to \sim Ew)$
3. ~~Show~~ $Dw \to \sim Ew$
4. Dw — ass cd
5. Show $\sim Ew$
6. Ew — ass id
7. $Dw \wedge Ew \to \exists u(Du \wedge Fw)$ — pr1 ui ← no capturing occurs here
8. $\exists u(Du \wedge Fw)$ — 4 6 adj 7 mp
9. $Ds \wedge Fw$ — 8 ei
10. Fw — 9 s
11. $Dw \to \sim Fw$ — pr2 ui
12. $\sim Fw$ — 4 11 mp 10 id
13. 3 ud

Because we used 'x' instead of 'u', we did not encounter any capturing problems in applying rule ui. Now we merely apply rule av to line 2, and we are done:

1. ~~Show~~ $\forall u(Du \to \sim Eu)$
2. ~~Show~~ $\forall w(Dw \to \sim Ew)$
3. ~~Show~~ $Dw \to \sim Ew$
4. Dw — ass cd
5. Show $\sim Ew$
6. Ew — ass id
7. $Dw \wedge Ew \to \exists u(Du \wedge Fw)$ — pr1 ui
8. $\exists u(Du \wedge Fw)$ — 4 6 adj 7 mp
9. $Ds \wedge Fw$ — 8 ei
10. Fw — 9 s
11. $Dw \to \sim Fw$ — pr2 ui
12. $\sim Fw$ — 4 11 mp 10 id
13. 3 ud
14. $\forall u(Du \to \sim Eu)$ — 2 av dd

EXERCISES

1. Provide derivations for these arguments.

a. $\sim\exists x(Ax \vee Bx)$
$\forall x\forall y(Gx \wedge Hy \to By)$
$\exists xGx$
$\therefore \forall x{\sim}Hx$

b. $\exists x(Hx \wedge \sim\exists y(Gy \wedge Hx))$
$\therefore \forall y{\sim}Gy$

c. $\forall x(Ax \to \forall y(Bx \leftrightarrow By))$
$\exists zBz$

$\therefore$ $\forall y(Ay \rightarrow By)$

d. $\sim\forall x(Dx \vee Ex)$
$\exists x(Fx \leftrightarrow \sim Ex) \rightarrow \forall zDz$
$\therefore$ $\exists x{\sim}Fx$

e. $Jc \wedge \sim Jd$
$\forall xKx \vee \forall x{\sim}Kx$
$\exists x(Jx \wedge Kx) \rightarrow \forall x(Kx \rightarrow Jx)$
$\therefore$ $\sim Kc$

2. Provide derivations for these theorems:

T229 $\exists x(\exists xFx \rightarrow Fx)$
T230 $\exists x(Fx \rightarrow \exists xFx)$
T234 $\forall x((Fx \rightarrow Gx) \wedge (Gx \rightarrow Hx) \rightarrow (Fx \rightarrow Hx))$
T235 $\forall x(Fx \rightarrow Gx) \wedge \forall x(Gx \rightarrow Hx) \rightarrow \forall x(Fx \rightarrow Hx)$
T236 $\forall x(Fx \leftrightarrow Gx) \wedge \forall x(Gx \leftrightarrow Hx) \rightarrow \forall x(Fx \leftrightarrow Hx)$
T237 $\forall x(Fx \rightarrow Gx) \wedge \forall x(Fx \rightarrow Hx) \rightarrow \forall x(Fx \rightarrow Gx \wedge Hx)$
T238 $\forall xFx \rightarrow \exists xFx$
T242 $\sim\forall x(Fx \rightarrow Gx) \leftrightarrow \exists x(Fx \wedge \sim Gx)$
T243 $\sim\exists x(Fx \wedge Gx) \leftrightarrow \forall x(Fx \rightarrow \sim Gx)$
T248 $\exists xFx \wedge \exists x{\sim}Fx \leftrightarrow \forall x\exists y(Fx \leftrightarrow \sim Fy)$

10 INVALIDITIES

In chapters 1 and 2 we studied tautological implication, which is formal validity that is due to how sentences are built up out of sentential letters and connectives. A sentence whose premises tautologically imply its conclusion is definitely valid. However, an argument may be valid even if its premises do not tautologically imply its conclusion if its validity is due to something in addition to how it is built up with connectives. We have seen examples of such arguments in this chapter, arguments such as:

$\forall xFx$
$\therefore$ Fa

In this chapter we have studied the kind of formal validity which is due to how formulas are built up out of names, monadic (one-place) predicates, variables, connectives and quantifiers. We call such validity "MPC validity" ("monadic predicate calculus validity"). Derivations using the methods of chapters 1-3 show that the arguments they validate are MPC valid. An argument which is MPC valid is definitely valid. Of course, an argument may be valid even if it is not MPC valid if its validity is due to something in addition to how it is built up from names, variables, monadic predicates, quantifiers, and connectives. Some examples of this are:

Some boy fed every cat <Uses the two-place predicate *'fed'*>
$\therefore$ *Every cat was fed by a boy*

There are infinitely many prime numbers <Uses the quantifier '*infinitely many*'>
$\therefore$ *There is at least one prime number.*

Dr. Jekyll is tall <Uses '*is*' in the sense of identity>
Dr. Jekyll is Mr. Hyde
$\therefore$ *Mr. Hyde is tall*

Still, even though MPC validity is not the whole story, it remains an important kind of validity.

So far in this chapter we have learned how to show that arguments are MPC valid by means of giving derivations which validate the arguments. We have not yet focused on how to show that an argument is not MPC valid. To do that we describe a logically possible situation in which, because of its MPC structure, the argument has true premises and a false conclusion. It is convenient in doing this to consider very "small" situations -- that is, situations in which only a small number of things exist. To illustrate this, suppose we are given this argument:

There are some fibers
Every fiber is green
Something isn't green
$\therefore$ *Everything green is a fiber*

Its MPC form is:

$\exists xFx$
$\forall x(Fx \rightarrow Gx)$
$\exists x{\sim}Gx$
$\therefore\ \forall x(Gx \rightarrow Fx)$

Now consider the following "small" situation:

There are three things:
The first is a fiber; the others are not.
The first and the second are green; the third is not.

In this situation the first premise, '$\exists xFx$', is true because the first thing is a fiber. The second premise, '$\forall x(Fx \rightarrow Gx)$', is true because there is only one fiber, and it is green. The third premise is true because something isn't green (the third thing). The conclusion is false because not everything that is green is a fiber -- the second thing is green but not a fiber. So this is a situation in which the argument has true premises and a false conclusion, and so it is not MPC valid.

If we reflect on this technique, we see that all that we need to show MPC invalidity is for this situation to have a certain kind of structure. We need that there be three things; that the first (and no other) is F, and that the first and second (but not the third) is G. Because this is enough to show that an argument of the given form isn't MPC valid, no matter whether 'F' means 'fiber', or 'feline', or 'tarantula, or whatever, and the same for 'G'. All that we need is that there is a situation with three things, of which the first is F and the first and second G. We need only say what there is in the "universe" of the situation, and which of these things are in the extensions of 'F' and 'G'; that is, which of these 'F' is true of, and which 'G' is true of. We will describe such a counter-example by using this format:

Universe: First thing Second thing Third thing

F: {the first thing}
G: {the first thing, the second thing}

This indicates how many things there are in the situation, and it gives the "extensions" of 'F' and 'G'. The extension of a predicate is just the set of things it is true of. So the information above tells us that 'F' is true of the first thing and of nothing else, and it tells us that 'G' is true of the first and second things, and not of the third.

Actually, to save on writing, we will just use some numbered icons, instead of 'first thing', 'second thing, etc:

Universe: ❶ ❷ ❸

F: {❶}
G: {❶, ❷}

This information describes a counter-example for the original argument, because it describes, in the most minimal terms, the structure of a situation in which the premises of the argument are true and the conclusion false.

Here are some more arguments that are not MPC valid, and counter-examples for them.

Counter-example #2:

$\exists x(Fx \wedge \sim Gx)$
$\forall x(Hx \rightarrow \sim Gx)$
$\exists x(Hx \wedge Fx)$
$\therefore\ \forall x(Fx \rightarrow \sim Gx)$

Universe: ❶ ❷ ❸

F: {❶, ❷}
G: {❶, ❸}
H: {❷}

The first premise is true because 'F' is true of ❷ and G isn't. The second premise is true because everything that 'H' is true of, namely ❷, 'G' is not true of, and the third premise is true because both 'F' and 'H' are true of ❷. But the conclusion is not true, because not everything that 'F' is true of is something that 'G' is not true of; ❶ is an example.

If the argument contains name letters, we indicate what they stand for in the given universe:

Counter-example #3:

$\forall x(Ax \rightarrow (Bx \leftrightarrow Cx))$
$Bk \wedge \sim Ck$
$\therefore\ \forall x \sim Ax$

Universe: ❶ ❷

A: {❷}
B: {❶, ❷}
C: {❷}
k: ❶ <'k' stands for the first thing>

The first premise is true because whatever 'A' is true of, namely ❷, is such that 'B' and 'C' are both true of it, so their biconditional comes out true. The second premise is true because 'B' is true of what 'k' stands for, namely ❶, and 'C' isn't. The conclusion is false because 'A' is not false of everything; it is true of ❷.

Counter-example #4:

$\forall x \exists y(Ax \leftrightarrow By)$
$\exists xBx \wedge \exists x \sim Bx$
$\forall x(Ax \rightarrow \sim Cx)$
$\therefore\ \sim \forall xCx$

Universe: ❶ ❷

A: { } <true of nothing at all>
B: {❶}
C: {❶, ❷}

The first premise is true because everything is such that something is such that 'A' is true of the first if and only if 'B' is true of the second. In fact, 'A' is true of nothing at all. And no matter what there is, there is something that 'B' is not true of, namely ❷. So there is always something that makes the biconditional true. The second premise is clearly true since 'B' is true of something, namely ❶, and 'B' is also false of something, namely ❷. The third premise is true because 'A' is true of nothing, so that every instance is a conditional with a false antecedent. The conclusion is false because 'C' is indeed true of everything.

Thinking up counter-examples: If you believe that an argument is not MPC valid, how do you think up a counter-example? There is a mechanical way to do this (described below), but it is too complex to be useful in many cases. So we will usually have to be creative. Still, some general observations may be useful in guiding our creativity. One approach that is often used is to build up the counter-example one piece at a time, guided by what is needed to make the premises true and conclusion false. Suppose we are given this argument:

$\forall x{\sim}(Fx \leftrightarrow Hx)$
$\exists x(Hx \wedge Gx)$
$\exists x(Hx \wedge {\sim}Gx)$
$\therefore\ \forall x(Fx \rightarrow Gx)$

So far, we don't know what will be in the universe. Begin by asking what is needed to make the conclusion false. In this case, what is needed is that there be something that 'F' is true of and 'G' is not. So write this:

F: {❶}
G: { } <not ❶>

The notation "<not ❶>" at the right is not part of the counter-example; it is merely a reminder to yourself that when constructing the counter-example you should not add ❶ to the list of things that 'G' is true of, because that could make the conclusion true.

Now consider the first premise; this says that whatever there is in the universe, 'F' and 'H' must disagree about it. This must be kept in mind as a constraint on what can be in the counter-example. So far, in fact, it tells us that since 'F' is true of , 'H' must not be:

F: {❶}
G: { } <not ❶>
H: { } <not ❶>

Next, consider the second premise: there is something that 'G' and 'H' are both true of. It can't be ❶, so fill in ❷:

F: {❶}
G: {❷} <not ❶>
H: {❷} <not ❶>

Next, the third premise; this says that there is something that 'H' is true of which 'G' is not true of. It can't be ❶ because 'H' cannot be true of ❶. It can't be ❷ because 'G' is true of ❷. So there must be a third thing:

F: {❶}
G: {❷} <not ❶>
H: {❷, ❸} <not ❶>

At this point we have all of the information we need. This is our proposed counter-example:

Universe: ❶ ❷ ❸

F: {❶}
G: {❷}
H: {❷, ❸}

If you check through the parts of the argument, you will see that the premises are all true and the conclusion false.

Sometimes if you start with no predicate being true of anything, a counter-example falls into your lap. Here is such a case. The argument is:

$\forall x(Jx \to Kx \vee Hx)$
$\sim\forall x(\sim Kx \to Jx)$
$\sim\exists x(Kx \wedge \sim Hx)$
$Hc \to \exists xJx$
$\therefore\ \sim\exists x(Hx \vee \sim Jx)$

Begin with this minimal proposed counter-example:

Universe: ❶

H: { }
J: { }
K: { }
c: ❶

Let us see what we need to add to what the predicates are true of to make this a counter-example. The first premise is already true because it is a quantified conditional with an antecedent that is false for each thing in the universe. The second premise is true because its unnegation '$\forall x(Kx \to \sim Jx)$' is false. This is false because the part following the quantifier: '$\sim Kx \to Jx$' is not true for every way of treating 'x' like a name; it is false when 'x' stands for ❶. The third is true because there is nothing that is K. The fourth is true because it is a conditional with a false antecedent. And the conclusion is false because there is indeed something that is either H or not J; ❶ is not J, so it is either H or not J. In short, the counter-example works as stated. (Usually, of course, more work will be needed.)

You may sometimes wonder how many things to put into the universe in order to produce a counter-example. There is no best way to determine this; usually you just put more things in when that seems to be required by the premises being true and the conclusion false. There is, however, an upper limit on what you need. If there is only one predicate letter in the argument, then you will need no more than two things in the universe. If there are two predicate letters, you will need no more than four. If there are three predicate letters, you will need no more than eight things. And so on. There is a formula for this: if there are n predicate letters, if there is a counter-example, there is one using no more than 2^n things.

Name letters have no effect on the number of things needed. If there are only two predicate letters, and thirteen name letters, then if there is a counter-example at all, there is one with four or fewer things. (Of course, if there are four things and thirteen name letters, several different constants will have to stand for the same things. But that's OK.)

So here is a mechanical way to come up with a counter-example. Decide, by the formula above, the maximum number of things needed in the universe for a counter-example. For example, suppose that there are two monadic predicates in the argument. The a universe of size 2^2, that is, 4, will do. Now just consider what choices the may be for the extension of predicate 'F'. There are 16 options:

{ }, {❶}, {❷}, {❸}, {❹}, {❶, ❷}, {❶, ❸}, {❶, ❹}, {❷, ❸}, {❷,❹}, {❸,❹}, {❶, ❷, ❸}, {❶, ❷, ❹},
{❶, ❸, ❹}, {❷, ❸, ❹}, {❶, ❷, ❸, ❹}

There are also 16 options for 'G'. So there are 16×16 = 256 options for possible counter-examples. If you just check these out, one at a time, you are sure to find one if one exists. If there are three monadic predicates there are 65,536 options. And so on.

(Exercise for the reader: In the above calculation we have supposed that if there is a counter-example, we can find one using a maximum size universe. We have ignored the possibility that there is, say, a counter-example using a universe of size 3 but none using a universe of size 4. Why are we justified in making that assumption?)

EXERCISES

1. Give counter-examples for each of the following arguments.

a. $\forall x(Ax \to \exists y(By \wedge \sim Ay))$
$\sim\forall xBx$
$\sim\exists x(Bx \wedge Cx)$
$\therefore\ \exists x(Ax \wedge Cx)$

b. $\exists x(Dx \wedge Ex \wedge \sim Fx)$
$\exists x(\sim Dx \wedge \sim Ex)$
$\forall x(Ex \to Dx \vee Fx)$
$\therefore\ \forall x(Dx \wedge Ex \to \sim Fx)$

c. $\exists x(Fx \wedge Gx)$
$\exists x(Fx \wedge \sim Gx)$
$\exists x(\sim Fx \wedge Gx)$
$\therefore\ \forall x(\sim Fx \to Gx)$ <requires more than three things in the universe>

d. $\forall x\exists y(Fx \leftrightarrow (Gy \vee Fx))$
$\therefore\ \sim\exists xFx \to \sim\exists xGx$

e. $Ha \wedge \sim Hb$
$\forall x(Kx \to Hx \wedge Jx)$
$\exists x(Jx \wedge \sim Kx)$
$\therefore\ \exists x(Hx \wedge \sim Jx)$

11 EXPANSIONS

In constructing counter-examples it is sometimes difficult to assess the truth value of a sentence in the counter-example, especially when it contains overlapping quantifiers. For example, ask yourself whether the following is a legitimate counter-example to this argument:

$\forall x\exists y(Ax \leftrightarrow {\sim}Ay)$
$\exists x(Ax \wedge Bx)$
$\therefore\ \forall xAx$

Universe: ❶ ❷ ❸

A: {❶}
B: {❶}

It is clear that this makes the conclusion false, and the second premise true. What about the first premise? It makes that true too. The first premise says that every thing in the universe is such that, there is a thing in the universe such that it isn't A if the first thing is A, and it is A if the first thing isn't. This is in fact true in the counter-example. But this may not be obvious to you. If not, there is a mechanical way to answer such a question. It resembles truth tables in that it will automatically give you a yes or no answer, but it may involve complexity. The technique is based on the idea that if there are a small number of things in the universe, then a universally quantified claim is equivalent to a conjunction of unquantified claims got by removing the quantifier and applying each resulting claim to a thing in the universe. And an existentially quantified claim, in turn, is equivalent to a disjunction of such claims that are applied to each thing in the universe.

Let us introduce a convention for naming things in a universe. When there are three things the names will be 'a_1', 'a_2', and 'a_3', where:

'a_1' stands for ❶
'a_2' stands for ❷
'a_3' stands for ❸.

(If there are fewer things, leave out 'a_3', or both 'a_2' and 'a_3'. If there are more things add 'a_4', 'a_5', and so on.) Now consider the sentence '$\forall xAx$'. This says that everything in the universe is A. This is equivalent to saying that the first thing is A *and* the second thing is A *and* the third thing is A. That is, it is equivalent to the conjunction:

$\forall xAx$ is equivalent to $Aa_1 \wedge Aa_2 \wedge Aa_3$

It is easy to check that this conjunction is true, because each conjunct is true.

The second premise is '$\exists x(Ax \wedge Bx)$'. This is equivalent to saying that either the first thing is both A and B, or the second thing is, or the third. That is, the quantified sentence is equivalent to this disjunction:

$\exists x(Ax \wedge Bx)$ is equivalent to $(Aa_1 \wedge Ba_1) \vee (Aa_2 \wedge Ba_2) \vee (Aa_3 \wedge Ba_3)$

It is easy to check that this disjunction is true, because at least one disjunct is true; the first disjunct is true.

The first premise, '$\forall x\exists y(Ax \leftrightarrow {\sim}Ay)$', is more interesting. It is universally quantified, so it is equivalent to the following conjunction:

$\exists y(Aa_1 \leftrightarrow {\sim}Ay) \wedge \exists y(Aa_2 \leftrightarrow {\sim}Ay) \wedge \exists y(Aa_3 \leftrightarrow {\sim}Ay)$

It may be easy to determine that this is true.

The first conjunct is true because there is something which is not A if and only if ❶ is A. We know that ❶ is A, and there is indeed at least one thing which is not A; for example, ❷ is not A.

The second conjunct is true because there is something which is not A if and only if ❷is A. We know that ❷ is not A, and there is indeed at least one thing which is A; for example, ❶ is A.

The third conjunct is just like the second; it is true because there is something which is not A if and only if ❸ is A. We know that ❸ is not A, and there is indeed at least one thing which is A; for example, ❶ is A.

Even this, however, is a bit subtle. There is a way to make it even more mechanical. Namely, each existentially quantified biconditional is equivalent to a disjunction. So this:

$$\begin{array}{l} \exists y(Aa_1 \leftrightarrow {\sim}Ay)\ \wedge \\ \exists y(Aa_2 \leftrightarrow {\sim}Ay)\ \wedge \\ \exists y(Aa_3 \leftrightarrow {\sim}Ay) \end{array}$$

is equivalent to this:

$$\begin{array}{l} (\,(Aa_1 \leftrightarrow {\sim}Aa_1) \vee (Aa_1 \leftrightarrow {\sim}Aa_2) \vee (Aa_1 \leftrightarrow {\sim}Aa_3)\,)\ \wedge \\ (\,(Aa_2 \leftrightarrow {\sim}Aa_1) \vee (Aa_2 \leftrightarrow {\sim}Aa_2) \vee (Aa_2 \leftrightarrow {\sim}Aa_3)\,)\ \wedge \\ (\,(Aa_3 \leftrightarrow {\sim}Aa_1) \vee (Aa_3 \leftrightarrow {\sim}Aa_2) \vee (Aa_3 \leftrightarrow {\sim}Aa_3)\,) \end{array}$$

And this is easy to evaluate. There are only three atomic sentences in this complex sentence: 'Aa_1', 'Aa_2', and 'Aa_3'. The first of these is true, and the others are false. It is thus easy to evaluate the biconditionals:

$$\begin{array}{l} (\,(Aa_1 \leftrightarrow {\sim}Aa_1) \vee (Aa_1 \leftrightarrow {\sim}Aa_2) \vee (Aa_1 \leftrightarrow {\sim}Aa_3)\,)\ \wedge \\ \quad\ \text{false} \qquad\qquad\qquad \text{true} \qquad\qquad\qquad \text{true} \end{array}$$

$$\begin{array}{l} (\,(Aa_2 \leftrightarrow {\sim}Aa_1) \vee (Aa_2 \leftrightarrow {\sim}Aa_2) \vee (Aa_2 \leftrightarrow {\sim}Aa_3)\,)\ \wedge \\ \quad\ \text{true} \qquad\qquad\qquad \text{false} \qquad\qquad\qquad \text{false} \end{array}$$

$$\begin{array}{l} (\,(Aa_3 \leftrightarrow {\sim}Aa_1) \vee (Aa_3 \leftrightarrow {\sim}Aa_2) \vee (Aa_3 \leftrightarrow {\sim}Aa_3)\,) \\ \quad\ \text{true} \qquad\qquad\qquad \text{false} \qquad\qquad\qquad \text{false} \end{array}$$

Each disjunction has a true disjunct, so each is true. So the conjunction of the disjunctions is also true. That is, the whole sentence, which is equivalent to '$\forall x\exists y(Ax \leftrightarrow {\sim}Ay)$', is true. This process is tedious, but completely mechanical.

If the counter-example has a universe of only one thing, then this device is easy to apply. Consider this argument and the accompanying counter-example:

$\forall x\forall y(Jx \leftrightarrow \exists z(Kz \leftrightarrow Jy))$
$\therefore\ \forall xJx$

Universe: ❶

J: { }
K: {❶}

It is clear that the conclusion is false, because 'J' is not true of ❶. The premise is universally quantified, so it is equivalent to a conjunction of all of its instances using names of things in the universe. Since there is only one thing in the universe, this conjunction has only one conjunct. So:

$\forall x\forall y(Jx \leftrightarrow \exists z(Kz \leftrightarrow Jy))$ is equivalent to $\forall y(Ja_1 \leftrightarrow \exists z(Kz \leftrightarrow Jy))$

But that in turn has a simpler equivalent:

$\forall y(Ja_1 \leftrightarrow \exists z(Kz \leftrightarrow Jy))$ is equivalent to $Ja_1 \leftrightarrow \exists z(Kz \leftrightarrow Ja_1)$

In '$Ja_1 \leftrightarrow \exists z(Kz \leftrightarrow Ja_1)$' the existentially quantified formula on the right is equivalent to a disjunction with only one disjunct, so we finally have:

$Ja_1 \leftrightarrow (Ka_1 \leftrightarrow Ja_1)$

The truth values of the parts of this sentence are:

'$Ja_1 \leftrightarrow (Ka_1 \leftrightarrow Ja_1)$'
false true false

and the whole sentence is true, as desired.

One more example. Consider the argument, and proposed counter-example:

$\forall x \exists y(Fx \lor Gy)$
$\sim\forall xFx$
$\sim\forall xGx$
$\therefore$ $\sim\exists xGx$

Universe: ❶ ❷

F: {❶}
G: {❷}

It is pretty clear that this proposed counter-example makes the conclusion false, since something is G, namely, ❷. The third premise is true since not everything is G; ❶ isn't G. Likewise, the second premise is true since not everything is F; ❷ is not F. What about the first? If you are not certain, you can expand it. In this proposed counter-example, the sentence '$\forall x \exists y(Fx \lor Gy)$', which starts with a universal quantifier, is equivalent to this conjunction:

$\exists y(Fa_1 \lor Gy) \land \exists y(Fa_2 \lor Gy)$

Each of the existentially quantified sentences is equivalent to a disjunction, so we have:

$((Fa_1 \lor Ga_1) \lor (Fa_1 \lor Ga_2)) \land ((Fa_2 \lor Ga_1) \lor (Fa_2 \lor Ga_2))$

evaluating the parts we have:

$((Fa_1 \lor Ga_1) \lor (Fa_1 \lor Ga_2)) \land ((Fa_2 \lor Ga_1) \lor (Fa_2 \lor Ga_2))$

$\underbrace{\text{true} \quad \text{false} \quad \text{true} \quad \text{true}}_{\text{true}} \qquad \underbrace{\text{false} \quad \text{false} \quad \text{false} \quad \text{true}}_{\text{true}}$

Each conjunct is true, so the sentence is itself true.

EXERCISES

1. For each of the following argument use the method of expansions to determine whether the following is a counterexample for it or not.

Universe: ❶ ❷ ❸

F: {❶}
G: {❶, ❸}
H: {❸}

a. $\forall x(Hx \rightarrow \exists y(Fy \land \sim Hy))$
$\sim\forall xFx$
$\sim\exists x(Fx \land Gx)$
$\therefore$ $\exists x(Hx \land Gx)$

b. $\exists x(Gx \land Hx \land \sim Fx)$
$\exists x(\sim Gx \land \sim Hx)$

$\forall x(Hx \to Gx \lor Fx)$
$\therefore \forall x(Gx \land Hx \to \sim Fx)$

c. $\exists x(Fx \land Gx)$
$\exists x(Fx \land \sim Gx)$
$\exists x(\sim Fx \land Gx)$
$\therefore \forall x(\sim Fx \to Gx)$

d. $\forall x \exists y(Fx \leftrightarrow (Gy \lor Fx))$
$\therefore \sim\exists xFx \to \sim\exists xGx$

e. $Ha \land \sim Hb$
$\forall x(Fx \to Hx \land Gx)$
$\exists x(Gx \land \sim Fx)$
$\therefore \exists x(Hx \land \sim Gx)$

BASIC RULES AND DERIVATION TECHNIQUES FOR CHAPTER 3

Rule ui: (universal instantiation):

$\forall$x ...x...x...	$\forall$x ...x...x...
∴ …b…b…	∴ …y…y…

Every occurrence of 'x' that '$\forall$x' was binding must be replaced with the same name or variable.
A new variable must not be introduced if some of its new occurrences are bound by a quantifier in the original formula.

Rule eg (existential generalization):

...b...b...	...y...y...
∴ $\exists$x…x…b…	∴ $\exists$x…x…b…

(You need not replace every occurrence of 'b' or of 'y' by 'x'.)
A new variable must not be introduced if some of its new occurrences are bound by a quantifier in the original formula.

Rule ei: (existential instantiation):

$\exists$x ...x...x...
∴ …y…y…

You must replace every occurrence of 'x' that '$\exists$x' was binding.
The variable 'y' must not already occur in the derivation or in a premise cited in the derivation..

Universal derivation:

If you have a derivation of the following form:

Show $\forall$x . . . x . . . x . . .
:::::
:::::
. . . x . . . x . . .

Then if there are no uncancelled show lines in between the first and last lines displayed, and if 'x' does not occur free anywhere in the derivation before the show line (or in a premise that has been cited in the derivation), you may box and cancel, using the notation 'ud'.

DERIVED RULES

Rule qn (Quantifier negation)

$\sim\forall xFx$	$\sim\exists xFx$	$\forall xFx$	$\exists xFx$
$\therefore\ \exists x\sim Fx$	$\therefore\ \forall x\sim Fx$	$\therefore\ \sim\exists x\sim Fx$	$\therefore\ \sim\forall x\sim Fx$
$\sim\forall x\sim Fx$	$\sim\exists x\sim Fx$	$\forall x\sim Fx$	$\exists x\sim Fx$
$\therefore\ \exists xFx$	$\therefore\ \forall xFx$	$\therefore\ \sim\exists xFx$	$\therefore\ \sim\forall xFx$

Rule av (alphabetic variance)

From a formula of the form '$\forall x \ldots x \ldots x \ldots$', where the initial quantifier has scope over the whole formula, you may infer '$\forall y \ldots y \ldots y \ldots$', which is the result of changing the variable 'x' in the quantifier to another variable, 'y', and changing all variables inside the first formula that are bound by the initial quantifier to 'y'.

Likewise if the initial quantifier is '$\exists$' instead of '$\forall$'.

Constraint: No capturing is allowed. That is, this inference is not permitted if the new variable becomes bound by a quantifier inside of the original formula.

STRATEGY HINTS

All of the strategy hints from chapters 1 and 2 still apply. These are new:

To derive:	Try this:
Universal Quantification **∀x□**	Set up a universal derivation. Write a show line containing ∀x□, and then immediately follow this with a show line containing □. When the second show is cancelled, use rule ud to cancel the first. Or write a show line with '∀x□', and then assume '~∀x□' for an indirect derivation. Turn this into '∃x~□', and proceed from there.
Existential Quantification **∃x□**	Derive an instance and then use rule eg. Or write a show line with '∀x□', and then assume '~∀x□' for an indirect derivation. Turn this into '∃x~□', and proceed from there.
Negation of a Universal Quantification **~∀x□**	State a show line with '~∀x□', and then assume '∀x□' for an indirect derivation. Or derive '∃x~□' and apply derived rule qn.
Negation of an Existential Quantification **~∃x□**	State a show line with '~∃x□', and then assume '∃x□' for an indirect derivation. Or derive '∀x~□' and apply derived rule qn.

If you have this available:	Try this:
Universal Quantification **∀x□**	Use rule ui to derive an instance. (But use rule ei first if that is an option.)
Existential Quantification **∃x□**	Use rule ei to derive an instance.
Negation of a Universal Quantification **~∀x□**	Use derived rule qn to turn this into an existential quantification.
Negation of an Existential Quantification **~∃x□**	Use derived rule qn to turn this into a universal quantification.

Use rule av if necessary: If you are having difficulty with capturing when you use rule ui or ei, change what you are trying to derive to an alphabetic variant. Complete the derivation, and then use derived rule av to convert this into a derivation of what you are after.

CHAPTER 3 THEOREMS

LAWS OF DISTRIBUTION:

T201 $\forall x(Fx \rightarrow Gx) \rightarrow (\forall xFx \rightarrow \forall xGx)$
T202 $\forall x(Fx \rightarrow Gx) \rightarrow (\exists xFx \rightarrow \exists xGx)$
T207 $\exists x(Fx \vee Gx) \leftrightarrow \exists xFx \vee \exists xGx$
T208 $\forall x(Fx \wedge Gx) \leftrightarrow \forall xFx \wedge \forall xGx$
T209 $\exists x(Fx \wedge Gx) \rightarrow \exists xFx \wedge \exists xGx$
T210 $\forall xFx \vee \forall xGx \rightarrow \forall x(Fx \vee Gx)$
T211 $(\exists xFx \rightarrow \exists xGx) \rightarrow \exists x(Fx \rightarrow Gx)$
T212 $(\forall xFx \rightarrow \forall xGx) \rightarrow \exists x(Fx \rightarrow Gx)$
T213 $\forall x(Fx \leftrightarrow Gx) \rightarrow (\forall xFx \leftrightarrow \forall xGx)$
T214 $\forall x(Fx \leftrightarrow Gx) \rightarrow (\exists xFx \leftrightarrow \exists xGx)$

LAWS OF QUANTIFIER NEGATION

T203 $\sim\forall xFx \leftrightarrow \exists x\sim Fx$
T204 $\sim\exists xFx \leftrightarrow \forall x\sim Fx$
T205 $\forall xFx \leftrightarrow \sim\exists x\sim Fx$
T206 $\exists xFx \leftrightarrow \sim\forall x\sim Fx$

LAWS OF CONFINEMENT

T215 $\forall x(P\wedge Fx) \leftrightarrow P\wedge\forall xFx$
T216 $\exists x(P\wedge Fx) \leftrightarrow P\wedge\exists xFx$
T217 $\forall x(P\vee Fx) \leftrightarrow P\vee\forall xFx$
T218 $\exists x(P\vee Fx) \leftrightarrow P\vee\exists xFx$
T219 $\forall x(P\rightarrow Fx) \leftrightarrow (P\rightarrow\forall xFx)$
T220 $\exists x(P\rightarrow Fx) \leftrightarrow (P\rightarrow\exists xFx)$
T221 $\forall x(Fx\rightarrow P) \leftrightarrow (\exists xFx\rightarrow P)$
T222 $\exists x(Fx\rightarrow P) \leftrightarrow (\forall xFx\rightarrow P)$
T223 $\forall x(Fx\leftrightarrow P) \rightarrow (\forall xFx\rightarrow P)$
T224 $\forall x(Fx\leftrightarrow P) \rightarrow (\exists xFx\rightarrow P)$
T225 $(\exists xFx\leftrightarrow P) \rightarrow \exists x(Fx\leftrightarrow P)$
T226 $(\forall xFx\leftrightarrow P) \rightarrow \exists x(Fx\leftrightarrow P)$

LAWS OF VACUOUS QUANTIFICATION

T227 $\forall xP \leftrightarrow P$
T228 $\exists xP \leftrightarrow P$
T229 $\exists x(\exists xFx \rightarrow Fx)$
T230 $\exists x(Fx \rightarrow \forall xFx)$

LAWS OF ALPHABETIC VARIANCE

T231 $\forall xFx \leftrightarrow \forall yFy$
T232 $\exists xFx \leftrightarrow \exists yFy$

OTHER

T233	$(Fx \to Gx) \wedge (Gx \to Hx) \to (Fx \to Hx)$
T234	$\forall x((Fx \to Gx) \wedge (Gx \to Hx) \to (Fx \to Hx))$
T235	$\forall x(Fx \to Gx) \wedge \forall x(Gx \to Hx) \to \forall x(Fx \to Hx)$
T236	$\forall x(Fx \leftrightarrow Gx) \wedge \forall x(Gx \leftrightarrow Hx) \to \forall x(Fx \leftrightarrow Hx)$
T237	$\forall x(Fx \to Gx) \wedge \forall x(Fx \to Hx) \to \forall x(Fx \to Gx \wedge Hx)$
T238	$\forall xFx \to \exists xFx$
T239	$\forall xFx \wedge \exists xGx \to \exists x(Fx \wedge Gx)$
T240	$\forall x(Fx \to Gx) \wedge \exists x(Fx \wedge Hx) \to \exists x(Gx \wedge Hx)$
T241	$\forall x(Fx \to Gx \vee Hx) \to \forall x(Fx \to Gx) \vee \exists x(Fx \wedge Hx)$
T242	$\sim\forall x(Fx \to Gx) \leftrightarrow \exists x(Fx \wedge \sim Gx)$
T243	$\sim\exists x(Fx \wedge Gx) \leftrightarrow \forall x(Fx \to \sim Gx)$
T244	$\sim\exists xFx \to \forall x(Fx \to Gx)$
T245	$\sim\exists xFx \leftrightarrow \forall x(Fx \to Gx) \wedge \forall x(Fx \to \sim Gx)$
T246	$\sim\exists xFx \wedge \sim\exists xGx \to \forall x(Fx \leftrightarrow Gx)$
T247	$\exists x(Fx \to Gx) \leftrightarrow \exists x\sim Fx \vee \exists xGx$
T248	$\exists xFx \wedge \exists x\sim Fx \leftrightarrow \forall x\exists y(Fx \leftrightarrow \sim Fy)$