22. The accompanying data describe flexural strength (Mpa) for concrete beams of a certain type was introduced in Example 1.2.

| 9.2 | 9.7 | 8.8 | 10.7 | 8.4 | 8.7 | 10.7 |
| 6.9 | 8.2 | 8.3 | 7.3  | 9.1 | 7.8 | 8.0  |
| 8.6 | 7.8 | 7.5 | 8.0  | 7.3 | 8.9 | 10.0 |
| 8.8 | 8.7 | 12.6 | 12.3 | 12.8 | 11.7 |

a. Calculate a point estimate of the mean value of strength for the conceptual population of all beams manufactured in this fashion, and state which estimator you used.  
\[ \bar{x} = \frac{\sum x_i}{n} = 9.1407 \]

b. Calculate a point estimate of the strength value that separates the weakest 50% of all such beams from the strongest 50%, and state which estimator you used.
\[ \tilde{x} = 8.7 \] (the middle observation when arranged in ascending order).

c. Calculate and interpret a point estimate of the population standard deviation \( \sigma \). Which estimator did you use?  
\[ s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = 1.6595 \]

d. Calculate a point estimate of the proportion of all such beams whose flexural strength exceeds 11 Mpa.  
\[ \hat{p} = \frac{s}{n} = \frac{4}{7} = .1481 \]

e. Calculate a point estimate of the population coefficient of variation \( \frac{\sigma}{\mu} \), and state which estimator you used.
\[ \hat{v} = \frac{s}{\bar{x}} = \frac{1.6595}{9.1407} = .1816 \]

ANSWER:

23. Answer the following questions.

a. A random sample of 10 houses in Big Rapids, each of which is heated with natural gas, is selected and the amount of gas (therms) used during the month of January is determined for each house. The resulting observations are 108, 161, 123, 94, 130, 152, 127, 114, 143, 104. Let \( \mu \) denote the average gas usage during January by all houses in this area. Compute a point estimate of \( \mu \).
\[ \hat{\mu} = \bar{x} = \frac{\sum x_i}{n} = 125.6 \]

b. Suppose there are 10,000 houses in Big Rapids that use natural gas for heating. Let \( \tau \) denote the total amount of gas used by all of these houses during January. Estimate \( \tau \) using the data of part (a). What estimator did you use in computing your estimate?

\[ \hat{\tau} = \frac{s}{\bar{x}} = \frac{1.6595}{9.1407} = .1816 \]

c. Use the data in part (a) to estimate \( p \), the proportion of all houses that used at least 105 therms.
\[ \hat{p} = \frac{s}{n} = \frac{4}{7} = .1481 \]

d. Give a point estimate of the population median usage (the middle value in the population of all houses) based on the sample of part (a). What estimator did you use?
\[ \hat{\mu} = \bar{x} = \frac{\sum x_i}{n} = 125.6 \]

ANSWER:
b. \( \hat{\mu} = 10,000 \) \( \hat{\mu} = 1,256,000 \)

c. 8 of 10 houses in the sample used at least 100 therms (the “successes”), so

\[ \hat{p} = \frac{8}{10} = .80 \]

d. The ordered sample values are 94, 104, 108, 114, 123, 127, 130, 143, 152, 161, from which the two middle values are 123 and 127, so

\[ \hat{\mu} = \bar{x} = \frac{123 + 127}{2} = 125.0 \]

24. Consider a random sample \( X_1, \ldots, X_n \) from the pdf

\[ f(x; \theta) = .5(1 + \theta x) \quad -1 \leq x \leq 1 \]

where \(-1 \leq \theta \leq 1\) (this distribution arises in particle physics). Show that \( \hat{\theta} = 3\bar{X} \) is an unbiased estimator of \( \theta \). [Hint: First determine \( \mu = E(X) = E(\bar{X}) \).]

**ANSWER:**

\[
E(X) = \int_{-1}^{1} x \cdot .5(1 + \theta x) \, dx = \left[ \frac{x^2}{2} + \frac{\theta x^3}{6} \right]_{-1}^{1} = \frac{1}{3} \theta \\
E(X) = \frac{1}{3} \theta, \quad E(\bar{X}) = \frac{1}{3} \theta \\
\hat{\theta} = 3\bar{X} \Rightarrow E(\hat{\theta}) = E(3\bar{X}) = 3E(\bar{X}) = 3\left( \frac{1}{3} \right) \theta = \theta 
\]

25. Let \( X_1, X_2, \ldots, X_n \) represent a random sample from a Rayleigh distribution with pdf

\[ f(x; \theta) = \frac{x}{\theta} e^{-x^2/(2\theta)} \quad x > 0 \]

a. It can be shown that \( E(X^2) = 2\theta \). Use this fact to construct an unbiased estimator of \( \theta \) based on \( \sum X_i^2 \) (and use rules of expected value to show that it is unbiased).

b. Estimate \( \theta \) from the following \( n = 10 \) observations on vibratory stress of a turbine blade under specified conditions:

<table>
<thead>
<tr>
<th>Observation</th>
<th>17.08</th>
<th>10.43</th>
<th>4.79</th>
<th>6.86</th>
<th>13.88</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observation</td>
<td>14.43</td>
<td>20.07</td>
<td>9.60</td>
<td>6.71</td>
<td>11.15</td>
</tr>
</tbody>
</table>

**ANSWER:**

a. \( E(X^2) = 2\theta \) implies that \( E\left( \frac{X^2}{2} \right) = \theta \). Consider \( \hat{\theta} = \sum \frac{X_i^2}{2n} \). Then

\[
E(\hat{\theta}) = E\left( \sum \frac{X_i^2}{2n} \right) = \frac{\sum E(X_i^2)}{2n} = \frac{\sum 2\theta}{2n} = \frac{2n\theta}{2n} = \theta, \quad \text{implying that } \hat{\theta} \text{ is an unbiased estimator for } \theta.
\]

b. \( \sum x_i^2 = 1535.7058 \), so \( \hat{\theta} = \frac{1535.7058}{20} = 76.785 \)
Section 6.2

26. A random sample of $n$ bike helmets manufactured by a certain company is selected. Let $X =$ the number among the $n$ that are flawed and let $p = P(\text{flawed})$. Assume that only $X$ is observed, rather than the sequence of $S$s and $F$s.

a. Derive the maximum likelihood estimator of $p$. If $n = 25$ and $x = 5$, what is the estimate?

b. Is the estimator of part (a) unbiased?

c. If $n = 25$ and $x = 5$, what is the mle of the probability $(1-p)^5$ that none of the next five helmets examined is flawed?

ANSWER:

a. We wish to take the derivative of $\ln\left[ \binom{n}{x} p^x (1 - p)^{n-x} \right]$, set it equal to zero and solve for $p$, yielding $\hat{p} = \frac{x}{n}$. For $n = 25$ and $x = 5$, $\hat{p} = 5/25 = 0.20$.

b. $E(\hat{p}) = E\left( \frac{X}{n} \right) = \frac{1}{n} E(X) = \frac{1}{n} (np) = p$; thus $\hat{p}$ is an unbiased estimator of $p$.

c. $(1 - 0.20)^5 = 0.3277$

27. Let $X$ denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose the pdf of $X$ is

$$f(x; \theta) = \begin{cases} (\theta + 1)x^\theta & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > -1$. A random sample of ten students yields data

$x_1 = 0.92, x_2 = 0.79, x_3 = 0.90, x_4 = 0.65, x_5 = 0.86, x_6 = 0.47, x_7 = 0.73, x_8 = 0.97, x_9 = 0.94, x_{10} = 0.77$.

a. Use the method of moments to obtain an estimator of $\theta$ and then compute the estimate for this data.

b. Obtain the maximum likelihood estimator of $\theta$ and then compute the estimate for the given data.

ANSWER:

a. $E(X) = \int_0^1 x(\theta + 1)x^\theta \, dx = \frac{\theta + 1}{\theta + 2} = 1 - \frac{1}{\theta + 2}$, so the moment estimator $\hat{\theta}$ is the solution to $\bar{X} - 1 = -\frac{1}{\hat{\theta} + 2}$, yielding $\hat{\theta} = \frac{1}{1 - \bar{X}} - 2$. Since $\bar{X} = 0.80, \hat{\theta} = 5 - 2 = 3$.

b. $f(x_1, \ldots, x_n; \theta) = (\theta + 1)^n x_1 x_2 \cdots x_n / \theta^n$, so the log likelihood is $n \ln(\theta + 1) + \theta \sum \ln(x_i)$. Taking $\frac{d}{d\theta}$ and equating to 0 yields $\frac{n}{\theta + 1} = -\sum \ln(x_i)$, so $\hat{\theta} = -\frac{n}{\sum \ln(x_i)} - 1$.

Taking $\ln(x_i)$ for each given $x_i$ yields ultimately $\hat{\theta} = 3.12$. 
28. The shear strength of each of ten test spot welds is determined, yielding the following data (psi):

<table>
<thead>
<tr>
<th>Strength (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>395</td>
</tr>
<tr>
<td>379</td>
</tr>
<tr>
<td>404</td>
</tr>
<tr>
<td>370</td>
</tr>
<tr>
<td>392</td>
</tr>
<tr>
<td>365</td>
</tr>
<tr>
<td>412</td>
</tr>
<tr>
<td>418</td>
</tr>
<tr>
<td>361</td>
</tr>
<tr>
<td>378</td>
</tr>
</tbody>
</table>

a. Assuming that shear strength is normally distributed, estimate the true average shear strength and standard deviation of shear strength using the method of maximum likelihood.
b. Again assuming a normal distribution, estimate the strength value below which 95% of all welds will have their strengths. (Hint: What is the 95 percentile in terms of $\mu$ and $\sigma$? Now use the invariance principle.)

**ANSWER:**

a. $\hat{\mu} = \bar{x} = 387.4; s^2 = 395.16$, so $\frac{1}{n} \sum (x_i - \bar{x})^2 = \hat{\sigma}^2 = \frac{9}{10} (395.16) = 355.64$
and $\hat{\sigma} = \sqrt{355.64} = 18.86$ (this is not $s$).
b. The 95th percentile is $\hat{\mu} + 1.645\hat{\sigma}$, so the mle of this is (by the invariance principle) is given by $\hat{\mu} + 1.645\hat{\sigma} = 418.42$.

29. Consider a random sample $X_1, X_2, \ldots, X_n$ from the shifted exponential pdf

$$f(x; \lambda, \theta) = \begin{cases} \lambda e^{-\lambda(x - \theta)} & x \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

a. Obtain the maximum likelihood estimators of $\theta$ and $\lambda$.
b. A random sample of size $n = 10$ results in the values 3.12, .65, 2.56, 2.21, 5.45, 3.43, 10.40, 8.94, 17.83, and 1.31, calculate the estimates of $\theta$ and $\lambda$.

**ANSWER:**

a. The joint pdf (likelihood function) is

$$f(x_1, \ldots, x_n; \lambda, \theta) = \begin{cases} \lambda^n e^{-\lambda \sum x_i - \lambda \theta} & x_1 \geq \theta, \ldots, x_n \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

Notice that $x_1 \geq \theta, \ldots, x_n \geq \theta$ iff $\min(x_i) \geq \theta$,
and that $-\lambda \sum (x_i - \theta) = -\lambda \sum x_i + n\lambda \theta$.
Thus likelihood $= f(x; \lambda, \theta) = \begin{cases} \lambda^n \exp(-\lambda \sum x_i) \exp(n\lambda \theta) & \min(x_i) \geq \theta \\ 0 & \min(x_i) < \theta \end{cases}$

Consider maximization wrt $\theta$. Because the exponent $n\lambda \theta$ is positive, increasing $\theta$ will increase the likelihood provided that $\min(x_i) \geq \theta$: if we make $\theta$ larger than $\min(x_i)$, the likelihood drops to 0. This implies that the mle of $\theta$ is $\hat{\theta} = \min(x_i)$. The log likelihood is now

$$n \ln(\lambda) - \lambda \sum (x_i - \hat{\theta})$$. Equating the derivative wrt $\lambda$ to 0 and solving yields $\hat{\lambda} = \frac{n}{\sum (x_i - \hat{\theta})} = \frac{n}{\sum x_i - n\hat{\theta}}$. 

46. Suppose that a random sample of 50 bottles of a particular brand of cough syrup is selected, and the alcohol content of each bottle is determined. Let \( \mu \) denote the average alcohol content for the population of all bottles of the brand under study. Suppose that the resulting 95% confidence interval is (8.0, 9.6).

a. Would a 90% confidence interval calculated from this same sample have been narrower or wider than the given interval? Explain your reasoning.

b. Consider the following statement: There is a 95% chance that \( \mu \) is between 8 and 9.6. Is this statement correct? Why or why not?

c. Consider the following statement: We can be highly confident that 95% of all bottles of this type of cough syrup have an alcohol content that is between 8.0 and 9.6. Is this statement correct? Why or why not?

d. Consider the following statement: If the process of selecting a sample of size 50 and then computing the corresponding 95% interval were repeated 100 times, 95 of the resulting intervals will include \( \mu \). Is this statement correct? Why or why not?

**ANSWER:**

a. A 90% confidence interval will be narrower. Also, the z critical value for a 90% confidence level is 1.645, smaller than the z of 1.96 for the 95% confidence level, thus producing a narrower interval.

b. Not a correct statement. Once an interval has been created from a sample, the mean \( \mu \) is either enclosed by it, or not. The 95% confidence is in the general procedure, for repeated sampling.

c. Not a correct statement. The interval is an estimate for the population mean, not a boundary for population values.

d. Not a correct statement. In theory, if the process were repeated an infinite number of times, 95% of the intervals would contain the population mean \( \mu \).

47. A CI is desired for the true average stray-load loss \( \mu \) (watts) for a certain type of induction motor when the line current is held at 10 amps for a speed of 1500 rpm. Assume that stray-load loss is normally distributed with \( \sigma = 3.0 \).

a. Compute a 95% CI for \( \mu \) when \( n = 25 \) and \( \bar{x} = 60 \).

b. Compute a 95% CI for \( \mu \) when \( n = 100 \) and \( \bar{x} = 60 \).

c. Compute a 99% CI for \( \mu \) when \( n = 100 \) and \( \bar{x} = 60 \).

d. Compute an 82% CI for \( \mu \) when \( n = 100 \) and \( \bar{x} = 60 \).

e. How large must \( n \) be if the width of the 99% interval for \( \mu \) is to be 1.0?

**ANSWER:**

a. \( 60 \pm \frac{1.96(3)}{\sqrt{25}} = 60 \pm 1.176 = (58.824, 61.176) \)

b. \( 60 \pm \frac{1.96(3)}{\sqrt{100}} = 60 \pm 0.588 = (59.412, 60.588) \)

c. \( 60 \pm \frac{2.58(3)}{\sqrt{100}} = 60 \pm 0.774 = (59.226, 60.774) \)

d. 82% confidence \( \Rightarrow 1 - \alpha = .82 \Rightarrow \alpha = .18 \Rightarrow \alpha/2 = .09 \), so \( z_{0.09} = z_{09} = 1.34 \) and the interval is \( 60 \pm \frac{1.34(3)}{\sqrt{100}} = 60 \pm 0.402 = 59.598, 60.402 \)
48. By how much must the sample size \( n \) be increased if the width of the CI \( (\bar{x} \pm \frac{z_{\alpha/2}\sigma}{\sqrt{n}}) \) is to be halved? If the sample size is increased by a factor of 25, what effect will this have on the width of the interval? Justify your assertions.

\( n = \frac{2(2.58)3^2}{1} = 239.62 \) so \( n = 240. \)

\( \bar{x} \) is the sample mean, \( \sigma \) is the standard deviation of the sample, \( z_{\alpha/2} \) is the critical value for the standard normal distribution.

**ANSWER:**

If \( L = 2z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \) and we increase the sample size by a factor of 4, the new length is

\[ L' = 2z_{\alpha/2} \frac{\sigma}{\sqrt{4n}} = \left[ 2z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right] \left( \frac{1}{2} \right) = \frac{L}{2}. \]

Thus halving the length requires \( n \) to be increased fourfold. If \( n' = 25n \), then \( L' = \frac{L}{5} \), so the length is decreased by a factor of 5.

49. Consider the 1000 95% confidence intervals (CI) for \( \mu \) that a statistical consultant will obtain for various clients. Suppose the data sets on which the intervals are based are selected independently of one another. How many of these 1000 intervals do you expect to capture the corresponding value of \( \mu \)? What is the probability that between 950 and 970 of these intervals contain the corresponding value of \( \mu \)? (Hint: Let \( Y \) be the number among the 1000 intervals that contain \( \mu \). What kind of random variable is \( Y \)?)

\( Y \) is a binomial r.v. with \( n = 1000 \) and \( p = .95 \), so \( E(Y) = np = 950 \), the expected number of intervals that capture \( \mu \), and \( \sigma_{\sqrt{n}p(1-p)} = 6.892 \). Using the normal approximation to the binomial distribution, \( P(950 \leq Y \leq 970) = P(949.5 \leq Y_{\text{normal}} \leq 970.5) = P(-.07 \leq Z \leq 2.97) = .9985 - .4721 = .5264 \).

**Section 7.2**

50. A random sample of 100 lightning flashes in a certain region resulted in a sample average radar echo duration of .81 sec and a sample standard deviation of .34 sec. Calculate a 99% (two-sided) confidence interval for the true average echo duration \( \mu \), and interpret the resulting interval.

**ANSWER:**

\[ \bar{x} \pm 2.58 \frac{s}{\sqrt{n}} = .81 \pm 2.58 \frac{.34}{\sqrt{100}} = .81 \pm .0877 = (.7223, .8977) \]

51. Determine the confidence level for each of the following large-sample one-sided confidence bounds:

a. Upper bound: \( \bar{x} + .84 \bar{x} + .84 s / \sqrt{n} \)

b. Lower bound: \( \bar{x} - 2.05 s / \sqrt{n} \)

c. Upper bound: \( \bar{x} + .67 s / \sqrt{n} \)
ANSWER:

a. \( z_a = .84 \), and \( \Phi(.84) = .7995 \approx .80 \), so the confidence level is 80%.

b. \( z_a = 2.05 \), and \( \Phi(2.05) = .9798 \approx .98 \), so the confidence level is 98%.

c. \( z_a = .67 \), and \( \Phi(.67) = .7486 \approx .75 \), so the confidence level is 75%.

52. It was reported that, in a sample of 507 adult Americans, only 142 correctly described the Bill of Rights as the first ten amendments to the U.S. Constitution. Calculate a (two-sided) confidence interval using a 99% confidence level for the proportion of all U.S. adults that could give a correct description of the Bill of Rights.

ANSWER:

\[
N = 507, \ x = \# \ of \ successes = 142 \Rightarrow \hat{p} = \frac{142}{507} = .28; \ the \ 99\% \ two-sided \ interval \ is:
\]

\[
\hat{p} \pm Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{.2866 \pm .0519}{1.0131} = (.2317,.3341)
\]

53. The superintendent of a large school district, having once had a course in probability and statistics, believes that the number of teachers absent on any given day has a Poisson distribution with parameter \( \lambda \). Use the accompanying data on absences for 50 days to derive a large-sample CI for \( \lambda \). [Hint: The mean and variance of a Poisson variable both equal \( \lambda \), so \( Z = (\bar{X} - \lambda)/\sqrt{\lambda/n} \) has approximately a standard normal distribution. Now proceed as in the derivation of the interval for \( p \) by making a probability statement (with probability \( 1 - \alpha \)) and solving the resulting inequalities for \( \lambda \).]

<table>
<thead>
<tr>
<th># Absences</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

ANSWER:

With \( \theta = \lambda, \hat{\theta} = \bar{X} \) and \( \sigma_{\theta} = \sqrt{\lambda/n} \) so \( \sigma_{\bar{X}} = \sqrt{\bar{X}/n} \). The large sample C.I. is then \( \bar{X} \pm z_{\alpha/2} \sqrt{\bar{X}/n} \). We calculate \( \sum x_i = 205 \), so \( \bar{X} = 4.16 \), and a 95% interval for \( \lambda \) is then given by:

\[
4.10 \pm 1.96 \sqrt{\frac{4.10}{50}} = 4.10 \pm .56 = (3.54, 4.66).
\]

Section 7.3

54. Determine the \( t \) critical value for a two-sided confidence interval in each of the following situations.

a. Confidence level = 95%, df = 12
b. Confidence level = 95%, df = 15
c. Confidence level = 99%, df = 20
d. Confidence level = 99%, n = 8
e. Confidence level = 98%, df = 25
f. Confidence level = 99%, n = 40
55. A random sample of \( n = 8 \) E-glass fiber test specimens of a certain type yielded a sample mean interfacial shear yield stress of 30.5 and a sample standard deviation of 3.0. Assuming that interfacial shear yield stress is normally distributed, compute a 95% CI for true average stress.

**ANSWER:**

\[ d.f. = n - 1 = 7, \text{ so the critical value for a 95\% C.I. is } t_{0.025,7} = 2.365. \]

The interval is then

\[
30.5 \pm (2.365) \left( \frac{3.0}{\sqrt{8}} \right) = 30.5 \pm 2.51(27.99, 33.01). 
\]

56. A sample of 14 joint specimens of a particular type gave a sample mean proportional limit stress of 8.50 MPa and a sample standard deviation of .80 MPa.

a. Calculate and interpret a 95\% lower confidence bound for the true average proportional limit stress of all such joints. What, if any, assumptions did you make about the distribution of proportional limit stress?

b. Calculate and interpret a 95\% lower prediction bound for the proportional limit stress of a single joint of this type.

**ANSWER:**

\[ n = 14, \bar{x} = 8.50, s = .80; t_{0.05,13} = 1.771 \]

a. A 95\% lower confidence bound:

\[
8.50 - 1.771 \left( \frac{.80}{\sqrt{14}} \right) = 8.5 - .379 = 8.121. 
\]

With 95\% confidence, the value of the true mean proportional limit stress of all such joints lies in the interval (8.11, \( \infty \)). If this interval is calculated for sample after sample, in the long run 95\% of these intervals will include the true mean proportional limit stress of all such joints. We must assume that the sample observations were taken from a normally distributed population.

b. A 95\% lower prediction bound:

\[
8.50 - 1.771(.80) \sqrt{1 + \frac{1}{14}} = 8.50 - 1.47 = 7.03. 
\]

If this bound is calculated for sample after sample, in the long run 95\% of these bounds will provide a lower bound for the corresponding future values of the proportional limit stress of a single joint of this type.

57. A study of the ability of individuals to walk in a straight line reported that accompanying data on cadence (strides per second) for a sample of \( n = 20 \) randomly selected healthy men:

\begin{align*}
.95 & \quad .81 & \quad .93 & \quad .95 & \quad .93 & \quad .86 & \quad 1.05 & \quad .92 & \quad .85 & \quad .81 \\
.92 & \quad .96 & \quad .92 & \quad 1.00 & \quad .78 & \quad 1.06 & \quad 1.06 & \quad .96 & \quad .85 & \quad .92
\end{align*}
A normal probability plot gives substantial support to the assumption that the population distribution of cadence is approximately normal. A descriptive summary of the data from MINITAB follows.

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
<th>SEMean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cadence</td>
<td>20</td>
<td>0.9255</td>
<td>0.9300</td>
<td>0.0809</td>
<td>0.0181</td>
</tr>
</tbody>
</table>

a. Calculate and interpret a 95% confidence interval for a population mean cadence.
b. Calculate and interpret a 95% prediction interval for the cadence of a single individual randomly selected from this population.
c. Calculate an interval that includes at least 99% of the cadences in the population distribution using a confidence level of 95%.

**ANSWER:**

a. A 95% C.I.: 
\[
.9255 \pm 2.093(.0181) = .9255 \pm .0379 \Rightarrow (.8876, .9634)
\]
b. A 95% P.I.: 
\[
.9255 \pm 2.093(.0809) \sqrt{\frac{1}{20}} = .9255 \pm .1735 \Rightarrow (.7520, 1.0990)
\]
c. A tolerance interval is requested, with \( k = 99 \), confidence level 95%, and \( n = 20 \). The tolerance critical value, from the table of “tolerance critical values for normal population distributions” available in your text, is 3.615. The interval is 
\[
.9255 \pm 3.615(.0809) \Rightarrow (.6330, 1.2180).
\]

58. A more extensive tabulation of \( t \) critical values than what appears in your text shows that for the \( t \) distribution with 20 df, the areas to the right of the values .687, .860, and 1.064 are .25, .20, and .15, respectively. What is the confidence level for each of the following three confidence intervals for the mean \( \mu \) of a normal population distribution? Which of the three intervals would you recommend be used, and why?

a. \((\bar{x} - .687s/\sqrt{n}, \bar{x} + 1.725s/\sqrt{n})\)
b. \((\bar{x} - .860s/\sqrt{n}, \bar{x} + 1.325s/\sqrt{n})\)
c. \((\bar{x} - 1.064s/\sqrt{n}, \bar{x} + 1.064s/\sqrt{n})\)

**ANSWER:**
The 20 d.f. row of the table of “critical values for \( t \) distribution” available in your text shows that 1.725 captures upper tail area .05 and 1.325 captures upper tail area .10. The confidence level for each interval is 100(central area)%.

For the first interval, central area = 1 – sum of tail areas = 1 - (.25 + .05) = .70, and for the second and third intervals the central areas are 1 - (.20 + .10) = .70 and 1 - (.15 + .15) = .70. Thus each interval has confidence level 70%. The width of the first interval is 
\[
\frac{s(1.687 + 1.725)}{\sqrt{n}} = \frac{2.412s}{\sqrt{n}},
\]
whereas the widths of the second and third intervals are 2.185 and 2.128 respectively. The third interval, with symmetrically placed critical values, is the shortest, so it should be used. This will always be true for a \( t \) interval.

Section 7.4

59. Determine the values of the following quantities:

a. \(\chi^2_{1,15}\)
b. \(\chi^2_{1,25}\)
c. \( \chi^2_{0.01,25} \)

d. \( \chi^2_{0.005,25} \)

e. \( \chi^2_{0.99,25} \)

f. \( \chi^2_{0.995,25} \)

**ANSWER:**

a. \( \chi^2_{1.15} = 22.307 \)

b. \( \chi^2_{2.25} = 34.381 \)

c. \( \chi^2_{0.025,22} = 2.180 \)

d. \( \chi^2_{0.05,25} = 46.925 \)

e. \( \chi^2_{0.25,22} = 11.523 \)

f. \( \chi^2_{0.95,25} = 10.519 \)

60. Determine the following:

a. The 90th percentile of the chi-squared distribution with \( \nu = 12 \).

b. The 10th percentile of the chi-squared distribution with \( \nu = 12 \).

c. \( P(10.98 < \chi^2 \leq 36.78) \) where \( \chi^2 \) is a chi-squared rv with \( \nu = 22 \).

d. \( P(\chi^2 < 14.611 \text{ or } \chi^2 > 37.652) \) where \( \chi^2 \) is a chi-squared rv with \( \nu = 25 \)

**ANSWER:**

a. \( \chi^2_{10.12} = 18.549 \)

b. \( \chi^2_{90.12} = 6.304 \)

c. Since \( 10.982 = \chi^2_{975.22} \) and \( 36.78 = \chi^2_{025.22} \), \( P(\chi^2_{975.22} \leq \chi^2 \leq \chi^2_{025.22}) = .95 \).

d. Since \( 14.61 = \chi^2_{95.25} \) and \( 37.65 = \chi^2_{05.25} \), \( P(\chi^2_{95.25} \leq \chi^2 \leq \chi^2_{05.25}) = .90 \).

61. The amount of lateral expansion (mils) was determined for a sample of \( n = 9 \) pulsed-power gas metal arc welds used in LNG ship containment tanks. The resulting sample standard deviation was \( s = 2.80 \) mils. Assuming normality, derive a 95% CI for \( \sigma^2 \) and for \( \sigma \).

**ANSWER:**

\( n - 1 = 8 \), \( \chi^2_{0.025,8} = 17.543 \), \( \chi^2_{0.975,8} = 2.180 \),

The 95% interval for \( \sigma^2 \) is then \( \left( \frac{8(7.84)}{17.543} \right) \left( \frac{87.84}{2.180} \right) = (3.575, 28.771) \).

The 95% interval for \( \sigma \) is \( \sqrt{3.575}, \sqrt{28.771} \) = (1.891, 5.364).

62. The results of a Wagner turbidity test performed on 15 samples of standard Ottawa testing sand were (in microamperes)

\begin{align*}
26.9 &\quad 25.8 &\quad 24.4 &\quad 24.1 &\quad 26.4 &\quad 25.9 &\quad 24.0 &\quad 21.7 \\
24.9 &\quad 25.9 &\quad 27.3 &\quad 26.7 &\quad 26.9 &\quad 24.8 &\quad 27.3
\end{align*}

a. Is it plausible that this sample was selected from a normal population distribution?
b. Calculate an upper confidence bound with confidence level 90% for the population standard deviation of turbidity.

**ANSWER:**

a. Using a normal probability plot, we ascertain that it is plausible that this sample was taken from a normal population distribution.

b. With $s = 1.579$, $n = 15$, and $\chi^2_{10.14} = 21.064$ the 90% upper confidence bound

For $\sigma$ is $\sqrt{\frac{14(1.579)^2}{21.064}} = 1.287$. 