Uniform Asymptotics of the Meixner Polynomials

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Introduction: Meixner polynomials

- For $\beta > 0$ and $0 < c < 1$, the Meixner polynomials are explicitly given by

$$M_n(z;\beta,c) = {}_2F_1\left(\begin{matrix}-n,-z\\ \beta\end{matrix}\middle|1-\frac{1}{c}\right) = \sum_{k=0}^{n}\frac{(-n)_k(-z)_k}{(\beta)_k k!}\left(1-\frac{1}{c}\right)^k,$$

where $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}^*$.

- The Meixner polynomials satisfy the discrete orthogonality condition

$$\sum_{k=0}^{\infty}\frac{c^k(\beta)_k}{k!}M_m(k;\beta,c)M_n(k;\beta,c) = \frac{c^{-n}n!}{(\beta)_n(1-c)^{\beta}}\delta_{mn}.$$

- Our problem is to find the large-n behavior of $M_n(z;\beta,c)$.

Introduction: background

- The Meixner polynomials have many applications in statistical physics. (Borodin-Olshanski CMP 2000, Johansson CMP 2000)

- Using probabilistic arguments, Maejima and Van Assche (Math. Proc. Cambridge Philos. Soc. 1985) have given an asymptotic formula for $M_n(n\alpha;\beta,c)$ when $\alpha<0$ and β is a positive integer. Their result is given in terms of elementary functions.

- Jin and Wong (Constr. Approx. 1998) have used the steepest-descent method for integrals to derive two infinite asymptotic expansions for $M_n(n\alpha;\beta,c)$. One holds uniformly for $0<\varepsilon\le\alpha\le 1+\varepsilon$, and the other holds uniformly for $1-\varepsilon\le\alpha\le M<\infty$; both expansions involve the parabolic cylinder function and its derivative.

Introduction: motivation of our study

Problem: Large-n behavior of $M_n(n\alpha; \beta, c)$.

- $\alpha < 0$ is okay.
- $0 < \varepsilon \le \alpha \le M$ is okay.
- What about $0 \le \alpha \le \varepsilon$ and $-\varepsilon \le \alpha \le 0$?

Introduction: difficulties in steepest-descent method

$$\frac{(\beta)_n}{n!} M_n(n\alpha - \beta/2; \beta, c) = \frac{e^{\mp(n\pi i\alpha - \beta\pi i/2)}}{2\pi i} \int_{-\infty}^{(0+)} \frac{\exp\{nf(w,\alpha)\}dw}{w(1-w)^{\beta/2}(w/c-1)^{\beta/2}},$$

where $f(w,\alpha) := \alpha \ln(w/c - 1) - \alpha \ln(1-w) - \ln w$.

- Turning points: $a = \frac{1-\sqrt{c}}{1+\sqrt{c}}$, $b = \frac{1+\sqrt{c}}{1-\sqrt{c}}$.
- Case $\alpha \in [a-\varepsilon, a+\varepsilon]$ or $\alpha \in [b-\varepsilon, b+\varepsilon]$: Airy function
- Case $\alpha \in [\varepsilon, a-\varepsilon] \cup [a+\varepsilon, b-\varepsilon] \cup [b+\varepsilon, \infty)$: elementary function
- Case $0 \le \alpha \le \varepsilon$?

Introduction: our approach

- In view of Gauss's contiguous relations for hypergeometric functions, we may restrict our study to the case $1 \leq \beta < 2$.

- Fixing any $0 < c < 1$ and $1 \leq \beta < 2$, we intend to investigate the large-n behavior of $M_n(nz - \beta/2; \beta, c)$ for z in the whole complex plane.

- Our approach is based on the Deift-Zhou steepest-descent method for oscillatory Riemann-Hilbert problems.

Introduction: Deift-Zhou steepest-descent method

- Deift and Zhou (Ann. of Math. 1993): modified KdV equation.
- Deift et al. (CPAM 1999): orthogonal polynomials with respect to exponential weights.
- Baik et al. (Annals of Mathematics Studies 2007): orthogonal polynomials with respect to a general class of discrete weights.

Methodology

- 1D $\rightarrow$ 2D (Fokas, Its and Kitaev): relate the Meixner polynomials with a 2×2 matrix-valued function which is the unique solution to an interpolation problem.

- Discrete $\rightarrow$ Continuous (Baik et al.): change the discrete interpolation problem to a continuous Riemann-Hilbert problem (RHP) whose unique solution can be expressed in terms of the solution to the basic interpolation problem.

- Deift-Zhou steepest-descent method: change the oscillate RHP to an equivalent RHP which can be asymptotically decomposed into several local RHPs.

- Global $\rightarrow$ Local (Deift et al.): decompose the global RHP into several local RHPs and choose some suitable local solutions such that these solutions can be pieced together to build a global approximate solution.

Step 1: 1D $\to$ 2D

Define

$$P(z) := \begin{pmatrix} \pi_n(z) & \sum_{k=0}^{\infty} \frac{\pi_n(k)w(k)}{z-k} \\ \gamma_{n-1}^2 \pi_{n-1}(z) & \sum_{k=0}^{\infty} \frac{\gamma_{n-1}^2 \pi_{n-1}(k)w(k)}{z-k} \end{pmatrix}.$$

For any $k \in \mathbb{N}$, we have

$$\operatorname*{Res}_{z=k} P_{12}(z) = \pi_n(k)w(k) = P_{11}(k)w(k),$$

$$\operatorname*{Res}_{z=k} P_{22}(z) = \gamma_{n-1}^2 \pi_{n-1}(k)w(k) = P_{21}(k)w(k).$$

Thus,

$$\operatorname*{Res}_{z=k} P(z) = \lim_{z\to k} P(z) \begin{pmatrix} 0 & w(z) \\ 0 & 0 \end{pmatrix}.$$

Step 2: Discrete $\rightarrow$ Continuous (example)

Suppose

$$\operatorname*{Res}_{z=0} \bar{Q}(z) = \lim_{z\to 0} \bar{Q}(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Define

$$\bar{R}(z) := \begin{cases} \bar{Q}(z) \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}, & \text{for any } z \in D(0,1) \setminus \{0\}; \\ \bar{Q}(z), & \text{for any } z \in \mathbb{C} \setminus \bar{D}(0,1). \end{cases}$$

We then have

$$\bar{R}_+(z) = \bar{R}_-(z) \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}, \quad \text{for any } z \in \partial D(0,1).$$

Step 3: Deift-Zhou steepest-descent method

- Mhaskar-Rakhmanov-Saff (MRS) numbers (turning points)

$$a = \frac{1-\sqrt{c}}{1+\sqrt{c}}, \qquad b = \frac{1+\sqrt{c}}{1-\sqrt{c}}.$$

- The equilibrium measure

$$\rho(x) = \begin{cases} \frac{1}{\pi}\arccos\frac{x(b+a)-2}{x(b-a)} & x \in [a,b]; \\ 1 & x \in [0,a]; \\ 0 & \text{otherwise.} \end{cases}$$

- Use the equilibrium measure to change the oscillate RHP to an equivalent RHP which can be asymptotically decomposed into several local RHPs.

Step 4: Global $\rightarrow$ Local

- Global two-dimensional Riemann-Hilbert problem is in general difficult to handle.

- Local two-dimensional Riemann-Hilbert problems are usually easy to solve, and the solutions are not unique.

- Decompose the global problem into several local problems and choose some suitable local solutions such that these solutions can be pieced together to build a global approximate solution.

Local problems near the turning points a and b

A.eps

The Airy parametrix was first introduced by Deift et al. (CPAM 1999).

Local problem near the band interval (a, b)

$$J_N(x) = \begin{cases} \begin{pmatrix} 0 & -(1-x)^{\beta-1} \\ (1-x)^{1-\beta} & 0 \end{pmatrix}, & \text{for any } \; x \in (a,1); \\ \\ \begin{pmatrix} 0 & -(x-1)^{\beta-1} \\ (x-1)^{1-\beta} & 0 \end{pmatrix}, & \text{for any } \; x \in (1,b). \end{cases}$$

$$N(z) = \begin{pmatrix} \dfrac{(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}} & \dfrac{-i(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}} \\ \\ \dfrac{i(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}} & \dfrac{(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}} \end{pmatrix}.$$

Local problem near the origin

(D1) $D(z)$ is analytic in $\mathbb{C} \setminus (-i\infty, i\infty)$;

(D2) $D_+(z) = D_-(z)[1 - e^{\pm 2i\pi(nz-\beta/2)}]$, for any $z \in (-i\infty, i\infty)$;

(D3) for $z \in \mathbb{C} \setminus (-i\infty, i\infty)$, $D(z) = 1 + O(|z|^{-1})$ as $z \to \infty$.

The solution is given by

$$D(z) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \left[\frac{\log(1-e^{-2n\pi s - i\pi\beta})}{s+iz} - \frac{\log(1-e^{-2n\pi s + i\pi\beta})}{s-iz}\right] ds\right\}.$$

As $n \to \infty$, $D(z) \sim 1$ uniformly for z bounded away from the origin.

Results: regions of approximation

rop.eps

The asymptotic formulas in the neighborhood of the origin involve the function $D(z)$.

Results: asymptotic formulas in a neighborhood of the origin

- For $z \in \Omega_l^0$, we have

$$\pi_n(nz-\beta/2) \sim D(z)n^n e^{ng(z)} \frac{(-z)^{(1-\beta)/2}(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(b-z)^{1/4}(a-z)^{1/4}}.$$

- For $z \in \Omega_{r,\pm}^0$, we have

$$\begin{aligned}\pi_n(nz-\beta/2) \quad &\sim \quad -2D(z)(-n)^n e^{nv(z)/2+nl/2}\sin(n\pi z-\beta\pi/2)e^{-n\widetilde{\phi}(z)} \\ &\times \frac{z^{(1-\beta)/2}(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(a-z)^{1/4}(b-z)^{1/4}}.\end{aligned}$$

Numerical evidence

	True value	Approximate value
$z = -1$	1.99529×10^{233}	1.99473×10^{233}
$z = -0.001$	8.36624×10^{187}	8.35137×10^{187}
$z = 0.001$	3.07930×10^{187}	3.07272×10^{187}
$z = 0.05$	-2.51701×10^{180}	-2.51507×10^{180}
$z = 0.171$	-9.12697×10^{174}	-9.12530×10^{174}
$z = 0.172$	-1.22035×10^{175}	-1.22003×10^{175}
$z = 2$	-4.71541×10^{201}	-4.70772×10^{201}
$z = 5.828$	2.78146×10^{259}	2.78231×10^{259}
$z = 5.829$	2.86933×10^{259}	2.87018×10^{259}
$z = 100$	2.16586×10^{399}	2.16586×10^{399}

The true values and approximate values of $\pi_n(nz - \beta/2)$ for $c = 0.5$, $\beta = 1.5$ and $n = 100$. Note that $a \approx 0.17157$ and $b \approx 5.82843$.

Conclusion: big hammer to strike a small nail

- We have used the Deift-Zhou steepest-descent method for oscillate RHP to derive uniform asymptotic formulas for the Meixner polynomials in a neighborhood of the origin.

- Can we use a simpler method, such as the ordinary steepest-descent method for integrals or Wang-Wong method for difference equations, to derive uniform asymptotic formulas for the Meixner polynomials in a neighborhood of the origin?

Discussions: asymptotic formulas near the origin

- Krawtchouk polynomials

  Integral technique: Qiu-Wong (Comput. Methods Funct. Theory 2004)

  RHP technique: Dai-Wong (Chin. Ann. Math. Ser. B 2007)

- Charlier polynomials

  ODE technique: Dunster (J. Approx. Theory 2001)

  RHP technique: Ou-Wong (to appear)

- Meixner polynomials

  Suggestion: try integral technique (Qiu-Wong), or ODE technique (Dunster), or difference technique (Olde Daalhuis), or difference-differential technique (Dominici).

Thank you!