Traveling Waves of Diffusive Predator-Prey Systems: Disease Outbreak Propagation

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Outline

- A 2-dim diffusive disease model
- Main theorem
  1. constructing convex invariant cone
  2. existence of traveling waves (Schauder fixed point theorem)
  3. non-existence of traveling waves (two-side Laplace transform)
- Extension to a 3-dim diffusive disease model
- Discussion
Reaction-Diffusion system

We study the following reaction-diffusion system in one-dimensional space.

\[
\begin{align*}
\partial_t S &= d_1 \partial_{xx} S - \beta SI/(S + I); \\
\partial_t I &= d_2 \partial_{xx} I + \beta SI/(S + I) - \gamma I
\end{align*}
\]

The basic reproduction number of the corresponding ODE system (Diekmann et al., 1990; van den Driessche & Watmough, 2002)

\[R_0 = \frac{\beta}{\gamma}.
\]

Minimal traveling speed

\[c^* = 2\sqrt{d_2(\beta - \gamma)}.
\]
Traveling wave solution pair

- Non-trivial traveling wave solutions \((S(x + ct), I(x + ct))\)

\[
\begin{align*}
    cS'' &= d_1S'' - \beta SI/(S + I); \\
    cI' &= d_2I'' + \beta SI/(S + I) - \gamma I.
\end{align*}
\]

- Boundary conditions

\[S(-\infty) = S_{-\infty}, S(\infty) = S_{\infty}, I(\pm\infty) = 0.\]

- \(S\) is non-increasing, \(I\) is non-negative and \(0 \leq S_{\infty} < S_{-\infty}\).
### Main theorem

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<th>conditions</th>
<th>non-trivial traveling wave</th>
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<tr>
<td>$R_0 &gt; 1$ and $c &gt; c^*$</td>
<td>existence</td>
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<tr>
<td>$R_0 \leq 1$ or $c &lt; c^*$</td>
<td>non-existence</td>
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Fixed point

- Traveling wave solution ($\alpha_1$ and $\alpha_2$ are sufficiently large)

\[
\Delta_1 S := -d_1 S'' + cS' + \alpha_1 S = \alpha_1 S - \beta SI/(S + I);
\]
\[
\Delta_2 I := -d_2 I'' + cI' + \alpha_2 I = \alpha_2 I + \beta SI/(S + I) - \gamma I.
\]

- Define $F = (F_1, F_2)$:

\[
F_1(S, I) := \Delta_1^{-1}[\alpha_1 S - \beta SI/(S + I)];
\]
\[
F_2(S, I) := \Delta_2^{-1}[\alpha_2 I + \beta SI/(S + I) - \gamma I].
\]

- We shall prove that $F$ has a fixed point $(S, I)$ in a suitable convex invariant cone.
Define $\Gamma$ be the set of all $(S, I)$ such that $S_- \leq S \leq S_+$ and $I_- \leq I \leq I_+$.

$$S_+(x) = S_{-\infty};$$
$$S_-(x) = \max\{S_{-\infty}(1 - M_1 e^{\varepsilon_1 x}), 0\};$$
$$I_+(x) = e^{\lambda x};$$
$$I_-(x) = \max\{e^{\lambda x}(1 - M_2 e^{\varepsilon_2 x}), 0\},$$

where $\lambda$ is the smaller root of the characteristic equation

$$f(\lambda) := -d_2 \lambda^2 + c \lambda - (\beta - \gamma) = 0.$$
Schauder fixed point theorem

• $F$ maps $\Gamma$ into $\Gamma$.

• $F$ is continuous and compact on $\Gamma$ with respect to the norm

$$|(\phi_1, \phi_2)|_\mu := \max\{\sup_{x \in \mathbb{R}} |\phi_1(x)| e^{-\mu|x|}, \sup_{x \in \mathbb{R}} |\phi_2(x)| e^{-\mu|x|}\},$$

where $\mu > 2\lambda$ is given.

• $F$ has a fixed point $(S, I)$ in $\Gamma$. 
Boundary conditions

• Squeeze lemma (sandwich rule) gives $S'(-\infty) = S_{-\infty}$ and $I(-\infty) = 0$.

• Integral representations of $\Delta_1^{-1}$ and $\Delta_2^{-1}$, together with L’hôpital’s rule yield $S'(-\infty) = 0$ and $I'(-\infty) = 0$.

• From differential equations we obtain $S''(-\infty) = 0$ and $I''(-\infty) = 0$. 
Boundary conditions

- Squeeze lemma (sandwich rule) gives $S'(-\infty) = S_{-\infty}$ and $I(-\infty) = 0$.

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- From differential equations we obtain $S''(-\infty) = 0$ and $I''(-\infty) = 0$.

- It can be proved that $S$ is non-increasing with $S(\infty) < S(-\infty)$, and $I(\infty) = 0$.

- Furthermore, we have $S'(\infty) = I'(\infty) = S''(\infty) = I''(\infty) = 0$. 
Non-existence of traveling waves

• If \( R_0 = \beta / \gamma > 1 \) and \( c < c^* := 2 \sqrt{d_2(\beta - \gamma)} \), then the characteristic equation
  \[ f(\mu) := -d_2 \mu^2 + c \mu + \beta - \gamma \]
is always negative on the real line.

• Taking two-sided Laplace transformation on
  \[-d_2 I'' + c I' - (\beta - \gamma) I = -\beta I^2 / (S + I)\]
yields
  \[ f(\mu) \int_{-\infty}^{\infty} e^{-\mu x} I(x) dx = \int_{-\infty}^{\infty} e^{-\mu x} \frac{-\beta I(x)^2}{S(x) + I(x)} dx. \]

• For large \( \mu \) we have \( f(\mu) + \frac{\beta I(x)}{S(x) + I(x)} \leq f(\mu) + \beta < 0 \)
  and
  \[ \int_{-\infty}^{\infty} e^{-\mu x} I(x) \left[ f(\mu) + \frac{\beta I(x)}{S(x) + I(x)} \right] dx < 0. \]
A further extension

We study the following 3-dim diffusive disease model

\[ \frac{\partial_t S}{1} = d_1 \partial_{xx} S - \frac{\beta SI}{S + I + R}; \]
\[ \frac{\partial_t I}{1} = d_2 \partial_{xx} I + \frac{\beta SI}{S + I + R} - \gamma I - \delta I; \]
\[ \frac{\partial_t R}{1} = d_3 \partial_{xx} R + \gamma I. \]

We look for traveling wave solutions of the form \((S(x + ct), I(x + ct), R(x + ct))\).

\[ cS' = d_1 S'' - \frac{\beta SI}{S + I + R}; \]
\[ cI' = d_2 I'' + \frac{\beta SI}{S + I + R} - (\gamma + \delta)I; \]
\[ cR' = d_3 R'' + \gamma I. \]
Main result

For any $S_{-\infty} > 0$, if $R_0 := \beta/(\gamma + \delta) > 1$, $c > c^* := 2\sqrt{d_2(\beta - \gamma - \delta)}$ and

$$d_3 < \frac{2d_2}{1 - \sqrt{1 - (c^*/c)^2}},$$

then there exist $S_{\infty} < S_{-\infty}$ and a traveling wave solution $(S, I, R)$ such that $S(-\infty) = S_{-\infty}$, $S(\infty) = S_{\infty}$, $I(\pm\infty) = 0$, $R(\infty) = \gamma(S_{-\infty} - S_{\infty})/(\gamma + \delta)$ and $R(-\infty) = 0$. Furthermore, $S(x)$ is decreasing, $0 \leq I(x) \leq S_{-\infty} - S_{\infty}$, $R(x)$ is increasing, and

$$\int_{-\infty}^{\infty} \gamma + \delta)I(x)dx = \int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x) + I(x) + R(x)}dx = c(S_{-\infty} - S_{\infty}).$$

On the other hand, if $c < c^*$ or $R_0 \leq 1$, then there does not exist a non-trivial and non-negative traveling wave solution $(S, I, R)$ such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$. 
Convex invariant cone

Construct the super- and sub-solutions:

\[
S_+(x) := S_{-\infty}; \quad S_-(x) := \max\{S_{-\infty}(1 - M_1 e^{\varepsilon_1 x}), 0\};
\]

\[
I_+(x) := e^{\lambda_0 x}; \quad I_-(x) := \max\{e^{\lambda_0 x}(1 - M_2 e^{\varepsilon_2 x}), 0\};
\]

\[
R_+(x) := \frac{\gamma e^{\lambda_0 x}}{c\lambda_0 - d_3 \lambda_0^2}; \quad R_-(x) := \max\{\frac{\gamma e^{\lambda_0 x}}{c\lambda_0 - d_3 \lambda_0^2}(1 - M_3 e^{\varepsilon_3 x}), 0\},
\]

where \(\lambda_0\) is the smaller root of the characteristic equation

\[
f(\lambda) := -d_2 \lambda^2 + c \lambda - (\beta - \gamma - \delta) = 0.
\]

Remark: we require an additional condition \(d_3 < \frac{c}{\lambda_0} = \frac{2d_2}{1 - \sqrt{1 - (c^*/c)^2}}.\)
Future work

• What is $S(\infty) = S_\infty$?

• Is the traveling wave solution unique?

• What happens for the limit case $R_0 > 1$ and $c = c^*$? The standard argument of taking limit $c \to c^*$ fails because $I$ is not monotone.

• What if the additional condition $d_3 < \frac{c}{\lambda_0} = \frac{2d_2}{1-\sqrt{1-(c^*/c)^2}}$ is violated?

• What if the model parameters are spatially periodic?
Thank you!